FPT Algorithms for Plane Completion Problems*

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Abstract

The Plane Subgraph (resp. Topological Minor) Completion problem asks, given a (possibly disconnected) plane (multi)graph Γ and a connected plane (multi)graph Δ, whether it is possible to add edges in Γ without violating the planarity of its embedding so that it contains some subgraph (resp. topological minor) that is topologically isomorphic to Δ. We give FPT algorithms that solve both problems in $f(|E(\Delta)|)\cdot|E(\Gamma)|^2$ steps. Moreover, for the Plane Subgraph Completion problem we show that $f(k) = 2^{\mathcal{O}(k\log k)}$.

1 Introduction

Completion problems on graphs are defined as follows: Consider a graph class $\mathcal{P}$ and ask whether we may add edges to a given graph G in order to obtain a graph G^+, where $G^+ \in \mathcal{P}$. Numerous results have appeared for the case where the objective is to minimize the number of edges added in G [16, 12, 14, 9, 3].

In this paper, we consider the Plane Subgraph (resp. Topological Minor) Completion (PSC) (resp. PTMC) problem which, given a (possibly disconnected) plane graph Γ, called the *host graph*, and a connected plane graph Δ, called the *pattern graph*, asks whether it is possible to add edges in Γ such that the resulting graph remains plane and contains some subgraph (resp. topological minor) that is topologically isomorphic to Δ. Both Γ and Δ are allowed to have multiple edges but not loops. When the input graph Γ is planar triangulated, both PSC and PTMC are NP-complete. Indeed, let G be any planar triangulated graph. Note here, that as any planar triangulated graph is 3-connected, G is 3-connected and from Whitney's Theorem [15] admits a unique embedding on the plane (up to equivalence), say Γ. Let also Δ be the cycle on $n = |V(G)|$ vertices. Then Δ also has unique embedding on the plane

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(up to equivalence). Since Γ is triangulated no edge can be added to it while preserving its planarity. Thus, both PSC and PTMC become equivalent to the Hamilton Cycle Problem which is NP-complete on planar triangulated graphs [5] (see also [8]). This observation further implies that PSC and PTMC parameterized by the number of added edges k, and in particular even for $k = 0$, are NP-complete. Thus, PSC and PTMC are not FPT when parameterized by the number of added edges unless P = NP. Thus, in order to obtain a tractable algorithm, we need to find an alternative way to parameterize these problems. In particular, we will consider $|E(\Delta)|$ as our parameter. Our two main results are the following.

Theorem. PSC *parameterized by the number of edges of the pattern graph Δ, say k, can be solved in $2^{\mathcal{O}(k \log k)} \cdot m^2$ time, where $m = |E(\Gamma)|$.*

Theorem. PTMC *parameterized by the number of edges of the pattern graph Δ, say k, can be solved in $f(k) \cdot m^2$ time, where $m = |E(\Gamma)|$ and f is a computable function.*

For the PTMC algorithm our approach is the following. Let Γ and Δ be an input of the problem as above. We first apply a series of transformations on our input graph Γ that turn it into a combinatorial structure $\mathbf{G}$ (while the topological properties of Γ are retained) whose treewidth is bounded by a function of $|E(\Delta)|$. Then, we apply a series of transformations on our input graph Δ that allow us to encode both the topological and combinatorial information of Δ using a combinatorial structure $\mathbf{D}$. Finally, we show that (Δ, Γ) is a yes-instance of our problem if and only if an MSO-expressible relation holds for $\mathbf{G}$ and $\mathbf{D}$, thus translating our problem into a purely combinatorial one. Then by employing Courcelle's Theorem we prove the existence of an algorithm for PTMC. We remark here that a similar approach could also solve the Plane Subgraph Completion problem. However, with a more careful analysis we are able to derive an algorithm which avoids the heavy parametric dependance (caused by the use of Courcelle's theorem) for the case of plane topological minors.

Our approach towards solving PSC is the following. Let Γ and Δ be an input of PSC, where $|E(\Delta)| = k$ for some positive integer k. We construct a family $\mathcal{G}$ consisting of $\mathcal{O}(n)$ combinatorial structures depending only on Γ whose underlying graphs have treewidth $\mathcal{O}(k)$. We also construct a family $\mathcal{H}$ consisting of $2^{\mathcal{O}(k \log k)}$ combinatorial structures depending only on Δ, again by applying a series of appropriate transformations on them (different than the transformations for PTMC). For the graphs Γ and Δ and the families $\mathcal{G}$ and $\mathcal{H}$, it holds that (Δ, Γ) is a yes-instance if and only if some structure $\mathbf{D} \in \mathcal{H}$ is contained as a *contraction* in a structure $\mathbf{G} \in \mathcal{G}$, denoted $\mathbf{D} \leq_c \mathbf{G}$. Therefore, we again translate our problem into one of combinatorial nature. Finally, for a fixed pair of structures $(\mathbf{D}, \mathbf{G}) \in \mathcal{H} \times \mathcal{G}$ with the above properties, we can decide in $2^{\mathcal{O}(k \log k)} \cdot m$ time whether $\mathbf{D} \leq_c \mathbf{G}$. Therefore, by testing for all pairs $(\mathbf{D}, \mathbf{G}) \in \mathcal{H} \times \mathcal{G}$ whether $\mathbf{D} \leq_c \mathbf{G}$, we decide in $2^{\mathcal{O}(k \log k)} \cdot m^2$ steps whether (Δ, Γ) is a yes-instance.

The paper is organized as follows. In Section 2 we give the necessary definitions. In Section 3 we present the algorithm for the PSC problem and in Section 4 we present the algorithm for the PTMC problem. In the concluding Section 5 we discuss about other completion problems that can be solved by modifying our results, such as the Plane Induced Subgraph Completion, the Plane Minor Completion, the Planar Rooted Topological Minor, and the Planar Disjoint Paths Completion problems.

2 Definitions

For a positive integer n, we denote $[n] = \{1, 2, \dots, n\}$. Given a set S, a near-partition of S is a family of sets S_1, S_2, $\dots$, S_k, where $S_i \cap S_j = \emptyset$, for every $i \neq j$, and $\cup_{i \in [k]} S_i = S$ (note that by the definition it is possible that $S_i = \emptyset$ for some $i \in [k]$). Unless stated otherwise, the graphs considered do not have loops but may have multiple edges. Given a graph G, we will denote by $V(G)$ the set of its vertices and $E(G)$ the set of its edges. We denote by $\mathbf{dist}_G(u, v)$ the distance of two vertices u and v in the graph G. Also, given a graph G, a vertex $u \in V(G)$, and $V_0 \subseteq V(G)$, we denote by $N_G(u)$ the neighborhood of u in G and by $N_G(V_0) := \bigcup_{v \in V_0} N_G(v) \setminus V_0$. Given a vertex v with exactly two neighbors v_1 and v_2, the dissolution of v is the operation where we delete v and add an edge $\{v_1, v_2\}$ (even if one existed already).

Let G be a graph. A subset S of its vertices is a *separator* of G if the graph $G - S := (V(G) \setminus S, E[V(G) \setminus S])$ is not connected. The size of a separator S is equal to $|S|$. The vertex contained in a separator of size 1 will be called a *cut-vertex*, while the vertices of a separator of size 2 will be called a *cut-pair*. For every integer $k > 1$, a graph G with at least $k + 1$ vertices is k-*connected* if G has no separators of size less than k. For definitions not explicitly stated on the paper as well as more details on general graphs, see [7].

We say that a graph is *plane* when it is embedded without crossings between its edges on the sphere $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. We treat a plane graph as its embedding in Σ, that is, we do not distinguish between a vertex of the graph and the point of the sphere used in the drawing to represent the vertex or between an edge and the open curve representing it. We often use the term "general graph" in order to stress that a graph is treated as a combinatorial structure and not as a topological (i.e., embedded) one. Also, given a plane graph Γ we use the term *general graph* of Γ to refer to Γ as a combinatorial structure. We use capital greek letters for plane graphs and capital latin letters for general graphs.

We denote by $\subseteq$, $\subseteq_{sp}$, $\subseteq_{in}$, $\leq_m$, and $\simeq$ the usual subgraph, spanning subgraph, induced subgraph, minor, and isomorphism relation between two graphs, respectively. Given a graph G and $V_0 \subseteq V(G)$, we denote by $G[V_0]$ the subgraph of G induced by V_0. We call V_0 *connected* if $G[V_0]$ is connected.

Let Γ be a plane graph and $u \in V(\Gamma)$. Then a tuple $(u_1, \ldots, u_k)$ (with possible repetitions) will be called a *cyclic neighborhood* of u, and will be denoted by $\mathcal{N}_\Gamma(u)$, if $\{u, u_1\}, \ldots, \{u, u_k\}$ are exactly the edges incident to u, as we meet them starting from (u, u_1) and proceeding clockwise.

Let A be a subset of $\mathbb{R}^n$. We define $\mathbf{int}(A)$ to be the interior of A, $\mathbf{cl}(A)$ its closure and $\mathbf{bd}(A) = \mathbf{cl}(A) \setminus \mathbf{int}(A)$ its border. Given a plane graph Γ we denote its *faces* by $F(\Gamma)$, i.e., $F(\Gamma)$ is the set of the connected components of $\Sigma \setminus \Gamma$ (in the operation $\Sigma \setminus \Gamma$ we treat Γ as the set of points of Σ corresponding to its vertices and its edges). Given a graph G we denote by $\mathcal{C}(G)$ the set of the connected components of G. For every $f \in F(\Gamma)$ we denote by $B_\Gamma(f)$ the graph induced by the vertices and edges of Γ whose embeddings are subsets of $\mathbf{bd}(f)$ and we call it the *boundary* of f. We also denote by $V(B_\Gamma(f))$ and $E(B_\Gamma(f))$ the vertices and the edges of $B_\Gamma(f)$, respectively.

We define a *closed walk* of a graph G to be a cyclic ordering $w = (v_1, \ldots, v_l, v_1)$ of vertices of $V(G)$ such that for any two consecutive vertices, say v_i, v_{i+1}, there is an edge between them in G, i.e., $\{v_i, v_{i+1}\} \in E(G)$. Note here that there may exist two distinct indices i, j such that $v_i, v_j \in w$ and $v_i = v_j$ (the walk can revisit a vertex). We will denote by $\ell_w = l$ the length of the respective closed walk w. We say that a walk w of a plane graph Γ is *facial* if there exist $f_i \in F(\Gamma)$ and $\Theta_j \in \mathcal{C}(B_\Gamma(f_i))$ such that the vertices of w are the vertices of $V(\Theta_j)$ and the cyclic ordering of w indicates the way these vertices are met when making a closed walk along Θ_j while always keeping f_i on the same side of the walk. Notice that a facial walk is unique (up to cyclic permutation).

Given that Γ is a plane graph and $\mathbf{w} = \{w_1, \ldots, w_p\}$ is a non-empty set of closed walks of Γ, we say that $\mathbf{w}$ is a *facial mapping* if there exists some face f of Γ such that $\mathcal{C}(B_\Gamma(f)) = \{\Theta_1, \Theta_2, \ldots, \Theta_p\}$ and w_j is a facial walk of Θ_j, $j \in [p]$. We define the length of the facial mapping $\mathbf{w}$ to be $\ell_{\mathbf{w}} = \sum_{i=1}^{p} \ell_{w_i}$. Given a plane graph Γ and $f \in F(\Gamma)$, we define $\mathbf{w}(f)$ as the facial mapping of Γ corresponding to f and define its length ℓ_f to be the length $\ell_{\mathbf{w}}(f)$ of its corresponding facial mapping. Observe that for every face $f \in \Gamma(F)$, its facial mapping $\mathbf{w}(f)$ is unique (up to permutations). Let C_1, C_2 be two disjoint closed curves of Σ. Let also D_i be the open disk of $\Sigma \setminus C_i$ that does not contain points of C_{3-i}, $i \in [2]$. The *annulus between* C_1 and C_2 is the set $\Sigma \setminus (D_1 \cup D_2)$ and we denote it by $A[C_1, C_2]$. Notice that $A[C_1, C_2]$ is a closed set.

Let Γ and Δ be two plane graphs. We say that Γ and Δ are *topologically isomorphic* if they are isomorphic via a bijection $g : V(\Gamma) \to V(\Delta)$ and there exists a function $h : F(\Gamma) \to F(\Delta)$, such that for every $f \in F(\Gamma)$, $g(\mathbf{w}(f)) = \mathbf{w}(h(f))$ (where $g(\mathbf{w}(f))$ is the result of applying g to every vertex of every closed walk in $\mathbf{w}$). We call the function $\alpha : V(\Gamma) \cup F(\Gamma) \to V(\Delta) \cup F(\Delta)$ such that $\alpha = g \cup h$, a *topological isomorphism* between Γ and Δ.

We say that a general graph G is *uniquely embeddable* if any two plane graphs Γ and Γ' that are embeddings of G in the sphere, are topologically isomorphic. We say that a plane graph Γ is *uniquely embedded* if its general graph G is uniquely embeddable, i.e., Γ is the unique embedding of G, up to topological isomorphism. Given two plane graphs Γ_1 and Γ_2 we say that they are the *same* graph if they are topologically isomorphic (and not just isomorphic).

Let Γ and Δ be two plane graphs and let $Z \subseteq V(\Gamma)$. We say that Δ is a *Z-embedded subgraph* of Γ, and write $\Delta \leq_{es}^{Z} \Gamma$, if Δ is topologically isomorphic to some subgraph of $\Gamma \setminus Z$. When $Z = \emptyset$, we say that Δ is an *embedded subgraph* of Γ and write $\Delta \leq_{es} \Gamma$.

Definition 1 (**Z-embedded topological minors**). *Let Γ and Δ be two plane graphs and let $Z \subseteq V(\Gamma)$. We say that Δ is a Z-*embedded topological minor *of Γ, and write $\Delta \leq_{etm}^{Z} \Gamma$ if there exist a function $\rho_1 : V(\Delta) \to V(\Gamma)$ and a function $\rho_2 : E(\Delta) \to \mathcal{P}(\Gamma)$, where $\mathcal{P}(\Gamma)$ denotes the set of all paths of Γ, such that*

1. *For every $v \in V(\Delta)$, $\rho_1(v) \notin Z$.*
2. *For every $e = \{u, v\} \in E(\Delta)$, the path $\rho_2(e)$ of Γ has $\rho_1(u)$ and $\rho_1(v)$ as its endpoints and if $e_1 \neq e_2$, then $\rho_2(e_1)$ and $\rho_2(e_2)$ are internally vertex-disjoint.*
3. *If $\Gamma\langle\rho_2\rangle$ is the graph obtained by the union of all paths in $\rho_2(E(\Delta))$ after we dissolve all vertices that are not vertices in $\rho_1(V(\Delta))$, then there is a topological isomorphism $\alpha : V(\Delta) \cup F(\Delta) \to V(\Gamma\langle\rho_2\rangle) \cup F(\Gamma\langle\rho_2\rangle)$ between Δ and $\Gamma\langle\rho_2\rangle$ where $\alpha|_{V(\Delta)} = \rho_1$.*

When $Z = \emptyset$, we just write $\Delta \leq_{etm} \Gamma$.

If in the 3rd condition of the above definition we replace topological isomorphism by isomorphism and consider general graphs, say H and G, we define the relation of H being a *Z-topological minor* of (G, Z).

For definitions regarding plane graphs not explicitly stated on the paper as well as more details on the subject, see [13].

2.1 Radial Enhancements

Let Γ be a plane graph. A *subdivided radial enhancement* of Γ is defined as a plane graph that can be constructed as follows: consider Γ, subdivide every edge once, add a vertex v_f inside each face f of Γ. Consider a permutation $(H_1, H_2, \ldots, H_s)$ of the connected components of $B_\Gamma(f)$ and a facial walk of each connected component. Then add edges connecting v_f with the vertices incident to $B_\Gamma(f)$ in such a way that the first vertices in the cyclic neighborhood of v_f are the vertices of H_1 and appear in the order of the fixed facial walk. Then the vertices of H_2 follow, etc. Observe that in the resulting embedding, every face that is incident to an edge of $E(\Gamma)$ is (planar) triangulated. This triangulation may have multiple edges unless the boundary of each face of Γ is a cycle, as can be seen in the two distinct examples of a subdivided radial enhancement of a disconnected plane graph Γ in Figure 1.

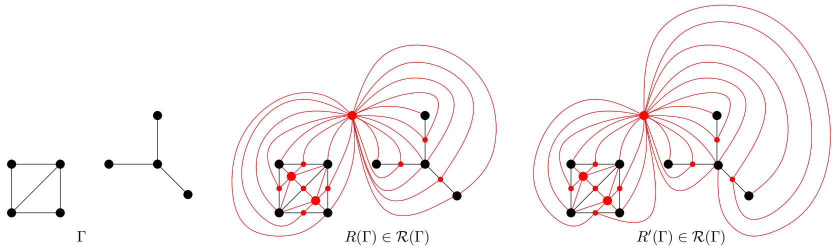

Figure 1: A disconnected plane graph Γ and two members of $\mathcal{R}_\Gamma$.

Notice that the vertices of the resulting plane graph can be partitioned into three independent sets: the *original vertices* of Γ denoted by $V_o(\Gamma)$, the *subdivision vertices* denoted by $V_s(\Gamma)$, which are the ones that were introduced after subdividing the edges, and the *radial vertices* denoted by $V_r(\Gamma)$, which are the ones that were added inside each face. Notice also that the edges of the resulting plane graph can be partitioned into two independent sets: the *subdivision edges* denoted by $E_s(\Gamma)$ and the *radial edges*, denoted by $E_r(\Gamma)$, that were introduced after adding the radial vertices.

We denote by $\mathcal{R}_\Gamma$ the set of all different (in terms of topological isomorphism) subdivided radial enhancements of Γ. Observe that if Γ is connected, then the boundary of each face of Γ is connected and we obtain the following.

Observation 1. *For every connected plane graph Γ, $R(\Gamma)$ is uniquely defined and thus $\mathcal{R}_\Gamma$ contains only one member.*

From the subdivided radial enhancement's construction we obtain the following.

Observation 2. *For every plane graph Γ and every $R(\Gamma) \in \mathcal{R}_\Gamma$ it holds that $|E(R(\Gamma))| = \mathcal{O}(|E(\Gamma)|)$.*

Given a plane graph Γ and a graph $R(\Gamma) \in \mathcal{R}_\Gamma$, for every integer $i > 1$, we denote by $R^i(\Gamma)$ the graph $R(R^{i-1}(\Gamma))$, where $R^1(\Gamma) = R(\Gamma)$. We define then $V_o^i(\Gamma) = V(R^{i-1}(\Gamma))$, $V_s^i = V_s(R^{i-1}(\Gamma))$, and $V_r^i = V_r(R^{i-1}(\Gamma))$. For notational consistency, we define $V_s^1(\Gamma) = V_s(\Gamma)$, $V_r^1(\Gamma) = V_r(\Gamma)$ and $V_o^1(\Gamma) = V_o(\Gamma)$. We also define the sets of edges $E_s^i(\Gamma)$ which are the edges obtained in $R^i(\Gamma)$ after subdividing the edges $E_s^{i-1}(\Gamma)$ and $E_r^i(\Gamma) = E(R^i(\Gamma)) \setminus E_s^i(\Gamma)$. Let Γ be a plane graph and $R(\Gamma)$ be a member of $\mathcal{R}_\Gamma$. For each vertex u of $R(\Gamma)$, we define the function $R^{-1} : V(R(\Gamma)) \to V(\Gamma) \cup E(\Gamma) \cup F(\Gamma)$ as follows:

- if $u \in V_o(R(\Gamma)) \equiv V(\Gamma)$ then $R^{-1}(u) = u$,
- if $u \in V_f(R(\Gamma))$ then $R^{-1}(u) = f_u$, where $f_u \in F(\Gamma)$ is the face in which u was added, and
- if $u \in V_s(R(\Gamma))$ then $R^{-1}(u) = e_u$, where $e_u \in E(G)$ is the edge whose subdivision created u.

We will call $R^{-1}(u)$ the *preimage* of u.

Lemma 1. *Let Γ be a plane graph (possibly with multiple edges). Then for every $R(\Gamma) \in \mathcal{R}_\Gamma$, it holds that $R^3(\Gamma)$ is 3-connected. Moreover, for every plane graph Γ each member of $\mathcal{R}_\Gamma$ is connected and if Γ is i-connected, then $R(\Gamma)$ is $(i+1)$-connected, for $i \in [2]$.*

In order to prove this lemma and only, we also need the following definitions as well as the next observation. Given a plane graph Γ, an *alternating face walk* is defined as a sequence of the form $(a_1, a_2, \dots, a_k)$, for $k > 1$, such that:

- for every $i \in [k-1]$, either $a_i \in V(\Gamma) \cup E(\Gamma)$, $a_{i+1} \in F(\Gamma)$, and a_i is in the boundary of a_{i+1}, or $a_i \in F(\Gamma)$, $a_{i+1} \in V(\Gamma) \cup E(\Gamma)$, and a_{i+1} is in the boundary of a_i,
- for every $i, j \in [k]$, $i \neq j \Rightarrow a_i \neq a_j$.

We will call a_1 and a_k the *endpoints* of the alternating face walk. The length of an alternating face walk $(a_1, a_2, \ldots, a_k)$ is k. The *alternating face distance* between two elements u, v of Γ is the smallest k for which there exists an alternating face walk of length k joining them.

The following observation follows directly from the definition of an alternating face walk.

Observation 3. *Let Γ be a plane graph, $R(\Gamma)$ be a subdivided radial enhancement of Γ, and $u, v \in V(R(\Gamma))$. Then $R(\Gamma)$ contains a (u, v)-path if and only if Γ contains an alternating face walk with endpoints $R^{-1}(u)$ and $R^{-1}(v)$.*

Proof of Lemma 1. The assertion that every member of $R(\Gamma)$ is connected follows immediately from Observation 3 together with the fact that a plane graph Γ always contains an alternating face walk between any pair $\{x, y\} \subseteq V(\Gamma) \cup E(\Gamma)$.

Suppose now that Γ is connected but $R(\Gamma)$ is not 2-connected and let $s \in V(R(\Gamma))$ be a cut-vertex of $R(\Gamma)$. Then s cannot be a radial vertex otherwise Γ would not be connected. Moreover, s cannot be a subdivision vertex, since the endpoints of the corresponding edge of Γ are connected to every neighbor of s that is a radial vertex (which is, at least one). Therefore s must be an original vertex of Γ and also, consequently, a cut-vertex of Γ. We choose a pair of vertices u, v in $V(R(\Gamma)) \setminus \{s\}$ in such a way that it satisfies the following conditions:

- If $u_\Gamma := R^{-1}(u)$ and $v_\Gamma := R^{-1}(v)$ are the preimages of u and v in Γ then every alternating face walk joining them in Γ contains s, and
- among all pairs satisfying the above condition they have the shortest alternating face distance.

Let us first assume that the shortest alternating face walk joining them has length $k > 3$. Let u'_Γ be the element of the alternating face walk that is the unique neighbor of u_Γ in this walk and assume that $u'_\Gamma \neq s$. Then u'_Γ and v_Γ are joined by an alternating face walk w of length $k-1$. Thus, from the choice of u and v, there exists an alternating face walk joining u'_Γ and v_Γ that does not contain s and hence we can trivially extend w' to an alternating face walk joining u_Γ and v_Γ that does not contain s, a contradiction. Thus $u'_\Gamma = s$. Let then v'_Γ be the unique neighbor of v_Γ in the alternating face walk. Then the alternating face walk joining u_Γ and v'_Γ contains s and has length $k-1$. From the choice of u, v there exists an alternating face walk w' joining u_Γ and v'_Γ that avoids s. Then, w' can trivially be extended to an alternating face walk joining u_Γ and v_Γ that avoids s. This is, again, a contradiction. Thus, any shortest alternating face walk joining u_Γ and v_Γ has length 3. This implies that the shortest alternating face walk between u_Γ and v_Γ is (u_Γ, s, v_Γ). Moreover, $u_\Gamma, v_\Gamma \in F(\Gamma)$. Then by utilising the edges that are incident to s and the faces f where $s \in \mathbf{bd}(f)$ we may find an alternating face walk joining u_Γ and v_Γ that avoids s, a contradiction. Therefore, $R(\Gamma)$ is 2-connected.

Suppose now that Γ is 2-connected but $R(\Gamma)$ is not 3-connected. Let then s, t be a cut-pair of $R(\Gamma)$ and s_Γ, t_Γ be their respective preimages in $R(\Gamma)$. Notice that neither s nor t is a radial vertex, as otherwise Γ would contain a separator of size less than 2. Again, as previously, we choose a pair of vertices u, v in $V(R(\Gamma)) \setminus \{s_\Gamma, t_\Gamma\}$ in such a way that it satisfies the following conditions:

- If $u_\Gamma := R^{-1}(u)$ and $v_\Gamma := R^{-1}(v)$ are the preimages of u and v in Γ then every alternating face walk joining them in Γ contains at least one of the s_Γ and t_Γ, and

- among all pairs satisfying the above condition they have the shortest alternating face distance.

We first show that if w is a shortest alternating face walk joining u_Γ, v_Γ then it contains exactly one of the s_Γ, t_Γ. Towards, a contradiction let us assume that both s_Γ and t_Γ belong to w. Then as $s_\Gamma, t_\Gamma \in V(\Gamma) \cup E(\Gamma)$ the elements do not appear consecutively in w. Let x_Γ be an element that appears between s_Γ and t_Γ in w. Then there exists an alternating face walk joining u_Γ and x_Γ and an alternating face walk joining x_Γ and v_Γ such that both of them have length strictly less than the length of w. From the choice of u and v, we obtain that there exist two alternative face walks w_u and w_v joining x_Γ with u_Γ and v_Γ respectively and such that none of them contains any of s_Γ, t_Γ. By combining them, we may obtain an alternating face walk joining u_Γ and v_Γ and containing neither s_Γ nor t_Γ, a contradiction. Thus w contains exactly one of the s_Γ and t_Γ, say s_Γ. Let us assume that w has length $k > 3$ and let u'_Γ be the element of the alternating face walk that is the unique neighbor of u_Γ in this walk and assume that $u'_\Gamma \neq s_\Gamma$. Then u'_Γ and v_Γ are joined by an alternating face walk w' of length $k-1$. Thus, from the choice of u and v, w' does not contain s_Γ and hence the alternating face walk joining u_Γ and v_Γ does not contain s_Γ or t_Γ, a contradiction. Thus $u'_\Gamma = s_\Gamma$. Let then v'_Γ be the unique neighbor of v_Γ in the alternating face walk w. Then the alternating face walk joining u_Γ and v'_Γ contains s_Γ and has length $k-1$. From the choice of u, v there exists an alternating face walk w' joining u_Γ and v'_Γ that avoids s_Γ. Then, w' can trivially be extended to an alternating face walk joining u_Γ and v_Γ that avoids s_Γ. This is a contradiction. Thus, any shortest alternating face walk joining u_Γ and v_Γ has length 3. Moreover, $w = (u_\Gamma, s_\Gamma, t_\Gamma)$. If $s_\Gamma \in V(\Gamma)$ then as above we may find an alternating face walk joining u_Γ and v_Γ that avoids s_Γ, a contradiction. Therefore, $s_\Gamma \in E(\Gamma)$. However, in that case, s_Γ belongs to the boundary of both faces u_Γ and v_Γ and at least one of the endpoints of s_Γ, say z_Γ, is different from t_Γ. Therefore, $(u_\Gamma, z_\Gamma, v_\Gamma)$ is an alternating face walk of Γ avoiding both s_Γ and t_Γ, a contradiction. We conclude that $R(\Gamma)$ is 3-connected. □

Remark 1. If Γ is 2-connected then $R(\Gamma)$ can also be shown to be 4-connected. However, 3-connectivity is sufficient for our purposes.

2.2 Graph Structures

A key-concept in our algorithms is the notion of the vertex and the edge structure which is formally defined as follows. Let G be a simple planar graph, $k, l \in \mathbb{N}$, $(S_1, S_2, \ldots, S_k)$ be a near-partition of $V(G)$ and $E_1, E_2, \ldots, E_l$ be a near-partition of $E(G)$. A *vertex structure* $\mathbf{G}$ is a tuple $(G, S_1, S_2, \ldots, S_k)$ and an edge structure $\mathbf{G}'$ is a tuple $(G, E_1, E_2, \ldots, E_l)$.

Let $\mathbf{G} = (G, A, X_1, \ldots, X_l)$ and $\mathbf{D} = (D, B, Y_1, \ldots, Y_l)$ be vertex structures, where $l \in \mathbb{N}$. We say that $\mathbf{D}$ is a *contraction* of $\mathbf{G}$, denoted by $\mathbf{D} \leq_c \mathbf{G}$, if and only if there exists a function $\sigma : V(G) \to V(D)$ satisfying the following *contraction properties*:

1. if $u, v \in V(D)$, $u \neq v \Leftrightarrow \sigma^{-1}(u) \cap \sigma^{-1}(v) = \emptyset$,
2. for every $u \in V(D)$, $G[\sigma^{-1}(u)]$ is connected,
3. $\{u, v\} \in E(D) \Leftrightarrow G[\sigma^{-1}(u) \cup \sigma^{-1}(v)]$ is connected,
4. $\sigma(A) \subseteq B$, and
5. for every $i \in [l]$ and every $x \in Y_i$ it holds that $|\sigma^{-1}(x)| = 1$ and $\sigma^{-1}(x) \in X_i$.

In particular, a graph D is a *contraction* of a graph G if $(D, V(D)) \leq_c (G, V(G))$ and we write $D \leq_c G$. Notice that $\leq_c$ defined for graphs is the usual contraction relation where only conditions 1, 2, and 3 apply. Observe that for any two vertex structures $\mathbf{G}$ and $\mathbf{D}$, where G and D respectively are their associated planar graphs, $\mathbf{D} \leq_c \mathbf{G}$ implies that $D \leq_c G$.

We will also need the following proposition, which follows from the results in [1].

Proposition 1. *There exists an algorithm that receives as input a vertex structure* $\mathbf{G}$*, whose graph has* m *edges and treewidth at most* h*, and a vertex structure* $\mathbf{D}$*, whose graph is connected and has* k *edges, and outputs whether* $\mathbf{D} \leq_c \mathbf{G}$ *in* $2^{\mathcal{O}(k+h+k\log h)} \cdot m$ *steps.*

Let $\mathbf{G} = (G, S_1, \ldots, S_l)$ be a vertex structure on a planar graph G, where $l \in \mathbb{N}$. Given a possibly empty $Q \subseteq V(G)$, notice that the tuple $(Q, S_1 \setminus Q, \ldots, S_l \setminus Q)$ also forms a near-partition of $V(G)$. Then, we can define the following operator on vertex structures:

$$\mathbf{d}(\mathbf{G}, Q) := (G, Q, S_1 \setminus Q, \ldots, S_l \setminus Q).$$

Obviously, $\mathbf{d}(\mathbf{G}, Q)$ is also a vertex structure on G.

Let Γ be a plane graph and consider an $R(\Gamma) \in \mathcal{R}_\Gamma$. By Lemma 1 and Observation 1, the graph $R^3(\Gamma)$ is uniquely defined according to $R(\Gamma)$. The following operators on $(\Gamma, R(\Gamma))$ uniquely define a vertex and an edge structure:

$$\begin{array}{rcl}\mathbf{p}(\Gamma, R(\Gamma)) & := & (R^3(\Gamma), V(\Gamma), V_s^1(\Gamma), V_r^1(\Gamma), V_s^2(\Gamma), V_r^2(\Gamma), V_s^3(\Gamma), V_r^3(\Gamma)) \\ \mathbf{e}(\Gamma, R(\Gamma)) & := & (R^3(\Gamma), E_s^3(\Gamma), E_r^3(\Gamma)).\end{array}$$

The underlying graph of the above structure is the general graph of $R^3(\Gamma)$ and the vertex sets that form the partition of $V(R^3(\Gamma))$ are the original vertices $V(\Gamma)$, followed by the sets of the subdivision and the radial vertices of each of the three subdivided radial enhancements. Moreover, the edges are separated to those that have been obtained in $R^3(\Gamma)$ only by subdividing original edges of the graph and those that where obtained after adding radial vertices and edges and subdividing those edges.

Throughout the rest of the paper we will only use structures defined by those three operators. The main purpose is to associate three subdivided radial enhancements to a given plane graph so that (i) the resulting graph is 3-connected and therefore uniquely embeddable, so we can disregard the embedding and treat it as a combinatorial object, and (ii) the vertices and edges of the original graph and each subdivided radial enhancement are distinguishable. In addition, both in PSC and the PTMC problems we try to match the faces of the pattern graph to faces, or parts of faces, of the host graph, the radial enhancements and the corresponding structures seem to be the appropriate tool to use, since we actually need to match just the radial vertices that are added inside each face.

Given a graph G and a non-negative integer k, we define the *ball* around a vertex v of G as the subgraph $B_G^k(v)$ of G induced by the set of vertices at distance at most k from v. Consider now the subgraph $\tilde{G}$ of G induced by the set of vertices that lay outside a given ball $B_G^k(v)$, i.e., $\tilde{G} = G \setminus B_G^k(v)$, and consider the set $\mathcal{C}(\tilde{G})$ of all its connected components. Then by contracting all the edges of every $C \in \mathcal{C}(\tilde{G})$ to a single vertex in G, denoted v_C, we obtain the *k-contracted graph* around v, that will be denoted by $G_v^{(k)}$. We can now make this contracted graph into a structure as follows. Given a vertex structure $\mathbf{G} = (G, \emptyset, S_1, \ldots, S_l)$ and a non-negative integer k, we define the *k-contracted vertex structure* around a vertex v of the graph G as $\mathbf{G}_v^{(k)} := (G_v^{(k)}, \{v_C \mid C \in \mathcal{C}(\tilde{G})\}, S'_1, \ldots, S'_l)$, where $S'_i = S_i \cap B_G^k(v)$ for every $i \in [l]$.

3 An FPT algorithm for the PSC problem

Given a plane graph Γ we define *the set of non-edges of* Γ: $\overline{E(\Gamma)} = \binom{V(\Gamma)}{2} \setminus E(\Gamma)$. A set of non-edges $S \subseteq \overline{E(\Gamma)}$ will be called *insertable* if there is a way to add the edges to Γ such that no two edges of $E(\Gamma) \cup S$ intersect (apart from any common endpoints). Finally, we define the following relation between two plane graphs Γ and Δ. We say that $\Delta \preceq \Gamma$ if there exists a set $S \subseteq \overline{E(\Gamma)}$ of insertable edges of Γ such that $\Delta \leq_{es} \Gamma'$, where Γ' is obtained from Γ after adding S. Then PSC asks, given two plane graphs Γ and Δ, whether $\Delta \preceq \Gamma$.

The main idea of our algorithm is to create two families of vertex structures, one from the host graph Γ and the other from the pattern graph Δ, such that $\Delta \preceq \Gamma$ if and only if there are two structures $\mathbf{D}$ and $\mathbf{G}$ from each of the above families such that $\mathbf{D} \leq_c \mathbf{G}$. Then, we bound the size of these families and use the algorithm from Proposition 1 to check all pairs of their members for the required property. From now on, in this section, whenever we refer to a structure we will assume that it is a vertex structure.

We define the first family of structures based on the host graph. Given a plane graph Γ, a subdivided radial enhancement of it, $R(\Gamma)$, and a positive integer k, we define the following family of structures:

$$\mathcal{G}_{\Gamma,R(\Gamma),k} := \{\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^{(k)} \mid v \in V(\Gamma)\}.$$

Obviously, $|\mathcal{G}_{\Gamma,R(\Gamma),k}| = |V(\Gamma)|$, regardless of the choice of $R(\Gamma)$ and k. In the following lemma we bound the treewidth of the underlying graphs of all members of this family.

Lemma 2. *Let Γ be a plane graph, $R(\Gamma)$ a subdivided radial enhancement of Γ, $k \in \mathbb{N}$, $v \in V(\Gamma)$, and $\mathbf{G}_v := \mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^{(k)} \in \mathcal{G}_{\Gamma,R(\Gamma),k}$. Then the underlying graph G_v of the structure $\mathbf{G}_v$ has treewidth at most $3(k+1)$ and size $\mathcal{O}(|E(\Gamma)|)$.*

Proof. Observe that the diameter of G_v is, by construction, at most $k+1$ and that G_v remains planar. From [4, Theorem 4] we obtain that G_v has treewidth at most $3(k+1)$. For the size, notice first that $G_v \equiv R^3(\Gamma)_v^{(k)}$. From Observation 2, $|E(R^3(\Gamma))| = \mathcal{O}(|E(\Gamma)|)$ and since $R^3(\Gamma)_v^{(k)} \leq_m R^3(\Gamma)$, it follows that $|E(G_v)| = \mathcal{O}(|E(\Gamma)|)$. □

In order to define the second family of structures based on the pattern graph we need the following two definitions.

A *facial extension* of a connected plane graph Δ is a connected plane graph Δ^+ satisfying the following properties:

1. $\Delta \subseteq \Delta^+$,
2. $V(\Delta^+) \setminus V(\Delta)$ is an independent set in Δ^+, and
3. for every distinct $x, y \in V(\Delta^+) \setminus V(\Delta)$, $N_{\Delta^+}(x) \nsubseteq N_{\Delta^+}(y)$.

We will denote by $\mathcal{F}_\Delta$ the family of all facial extensions of the graph Δ.

Given a connected plane graph Δ and a subset $L \subseteq E(\Delta)$ of its edges, we denote by $\mathsf{span}(\Delta, L)$ the set of all spanning subgraphs of Δ that contain all the edges in $E(\Delta) \setminus L$. Note that such subgraphs could also contain some edges in L. A *pattern-guess* of a connected plane graph Δ is an element Δ^* of $\mathsf{span}(\Delta^+, E(\Delta))$, for $\Delta^+ \in \mathcal{F}_\Delta$. That is, a spanning subgraph of a facial extension Δ^+ of Δ containing at least all the edges in $E(\Delta^+) \setminus E(\Delta)$. The family of all possible pattern-guesses Δ^* of Δ will be denoted by $\mathcal{PG}_\Delta$.

Now, given a connected plane graph Δ we define the following family of structures:

$$\mathcal{H}_\Delta := \{\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), V(\Delta^*) \setminus V(\Delta)) \mid \Delta^* \in \mathcal{PG}_\Delta,\ R(\Delta^*) \in \mathcal{R}_{\Delta^*}\}.$$

In the following lemma we bound the size of this second family and also the size and diameter of the underlying graphs of all members of the family.

Lemma 3. *If Δ is a connected plane graph then $|\mathcal{H}_\Delta| = 2^{\mathcal{O}(|E(\Delta)|\cdot\log|E(\Delta)|)}$ and, for any structure $\mathbf{D} \in \mathcal{H}_\Delta$, the underlying graph D of $\mathbf{D}$ has size and diameter bounded by $\mathcal{O}(|E(\Delta)|)$.*

Proof. First, note that $D \equiv R^3(\Delta^*)$, for some $\Delta^* \in \mathcal{PG}_\Delta$. Then, recall that since Δ^* is a spanning subgraph of a facial extension Δ^+ of Δ, the edges of Δ^* consist of the vertices of Δ^+ and some of the edges of Δ^+. By the construction of Δ^+, in each face of Δ the number of added vertices is a linear function of the size of the boundary of the face. Therefore, since Δ^+ is plane, $|E(\Delta^+)|$ is a linear function of $|E(\Delta)|$. Thus, $|E(\Delta^*)| = \mathcal{O}(|E(\Delta)|)$ and from Observation 2 we get $|E(R^3(\Delta^*))| = \mathcal{O}(|E(\Delta^*)|) = \mathcal{O}(|E(\Delta)|)$. Obviously, since $R^3(\Delta^*)$ is connected, $\mathbf{diam}(R^3(\Delta^*)) \leq |E(R^3(\Delta^*))| = \mathcal{O}(|E(\Delta)|)$.

Second, note that $|\mathcal{H}_\Delta| \leq |\mathcal{PG}_\Delta| \cdot \max\{|\mathcal{R}_{\Delta^*}| \mid \Delta^* \in \mathcal{PG}_\Delta\}$. If Δ^* is disconnected, then $\mathcal{R}_{\Delta^*}$ may contain at most $\mathcal{O}(|E(\Delta)|!) = 2^{\mathcal{O}(|E(\Delta)|\cdot\log|E(\Delta)|)}$ members. It is easy to see why this holds. In the construction of the subdivided radial enhancement, inside each face there will be created regions linear to the number of edges that are adjacent to that face. Then, the different connected components that lie in that face can be placed inside any of these regions and more edges will be added, equal to size of the facial walks around the connected components (i.e, up to twice the size of the boundary of the connected components). All these can be bounded by a factorial function of the size of the boundary of the face and since the number of faces of the graph is linear to the number of edges, the above bound is reached.

A pattern-guess Δ^* of $\mathcal{PG}_\Delta$ is constructed by first choosing a facial extension $\Delta^+ \in \mathcal{F}_\Delta$. Now, observe that in order to enumerate all the possible facial extensions of Δ, we can restrict ourselves to the enumeration of the facial extensions of a single face F of size q as, due to planarity, the sum of the degrees of its faces is a linear function of the number of its edges. Each facial extension of the face F can be constructed by adding at most q vertices inside F in layers, such that in each layer the neighborhoods of the vertices are non-crossing partitions of the boundary of F, where a *non-crossing partition* P of a set S is a partition with the extra property that if u_1, u_2, u_3, u_4 are vertices (not necessarily successive but strictly in that order) of the facial mapping of F, then $\forall S_1, S_2 \in S$ if $\{u_1, u_3\} \subseteq S_1$ and $u_2 \in S_2$ then $u_4 \notin S_2$. Since the non-crossing partitions of a set of size q can be bounded by the q-th Catalan number, i.e., by $\mathcal{O}(4^q) = 2^{\mathcal{O}(q)}$, the total number of the facial extensions of Δ can be bounded by $2^{\mathcal{O}(|V(\Delta)|\cdot\log|V(\Delta)|)} = 2^{\mathcal{O}(|E(\Delta)|\cdot\log|E(\Delta)|)}$, since Δ is connected. Therefore $|\mathcal{F}_\Delta| = 2^{\mathcal{O}(|E(\Delta)|\cdot\log|E(\Delta)|)}$. After choosing Δ^+, we then select an element of $\mathbf{span}(\Delta^+, E(\Delta))$, i.e., we choose and remove a subset of $E(\Delta)$ from Δ^+. This way we can obtain any member of $\mathcal{PG}_\Delta$. Hence, we conclude that

$$|\mathcal{PG}_\Delta| = 2^{|E(\Delta)|} \cdot 2^{\mathcal{O}(|E(\Delta)|\cdot\log|E(\Delta)|)} = 2^{\mathcal{O}(|E(\Delta)|\cdot\log|E(\Delta)|)}$$

and thus $|\mathcal{H}_\Delta| = 2^{\mathcal{O}(|E(\Delta)|\cdot\log|E(\Delta)|)}$. □

The next two lemmata will lead us to Theorem 1 which ensures the correctness of our algorithm.

Lemma 4. *Let Γ be a plane graph and Δ be a connected plane graph. If for every $R(\Gamma) \in \mathcal{R}_\Gamma$ there exists a positive integer c and two structures $\mathbf{G} \in \mathcal{G}_{\Gamma,R(\Gamma),c}$ and $\mathbf{D} \in \mathcal{H}_\Delta$, such that $\mathbf{D} \leq_c \mathbf{G}$, then $\Delta \preceq \Gamma$.*

Proof. Suppose now that for every $R(\Gamma) \in \mathcal{R}_\Gamma$ there exist two structures $\mathbf{G} \in \mathcal{G}_{\Gamma,R(\Gamma),c}$ and $\mathbf{D} \in \mathcal{H}_\Delta$, such that $\mathbf{D} \leq_c \mathbf{G}$. This is the same as saying that for every $R(\Gamma) \in \mathcal{R}_\Gamma$ there exist a $\Delta^+ \in \mathcal{F}_\Delta$, a $\Delta^* \in \mathsf{span}(\Delta^+, E(\Delta))$, and an $R(\Delta^*) \in \mathcal{R}_{\Delta^*}$ such that $\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B)$

$\leq_c \mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)$. Let then Γ' be the plane graph that results from $R^3(\Gamma)$ if we contract all connected components of $R^3(\Gamma)[\sigma^{-1}(B)]$. It follows immediately that $\Gamma' \simeq_{tp} R^3(\Delta^*)$. Let

$$\alpha : V(\Gamma') \cup F(\Gamma') \to V(R^3(\Delta^*)) \cup F(R^3(\Delta^*))$$

be a topological isomorphism between Γ' and $R^3(\Delta^*)$. Then, for each edge $\{u, v\} \in E(\Delta)\setminus E(\Delta^*)$ there is a face $f \in F(\Gamma)$ such that both $\alpha^{-1}(u)$ and $\alpha^{-1}(v)$ belong to a member of the facial mapping of f. Hence, the set $S = \big\{\{\alpha^{-1}(u), \alpha^{-1}(v)\} \mid \{u, v\} \in E(\Delta) \setminus E(\Delta^*)\big\}$ is insertable in Γ. Hence, $\Delta \preceq \Gamma$. $\square$

Lemma 5. *Let Γ be a plane graph and Δ be a connected plane graph. If $\Delta \preceq \Gamma$, then for every $R(\Gamma) \in \mathcal{R}_\Gamma$ there exist two structures $\mathbf{G} \in \mathcal{G}_{\Gamma,R(\Gamma),c}$ and $\mathbf{D} \in \mathcal{H}_\Delta$, such that $\mathbf{D} \leq_c \mathbf{G}$, where c is a constant such that $\max\limits_{\Delta^*}\{\mathbf{diam}(R^3(\Delta^*))\} \leq c$.*

To prove Lemma 5 we need the following lemma, which asserts the following: given the subdivided radial enhancements of the host and pattern graphs, we can restrict our search for the pattern graph to subgraphs of the host graph of bounded diameter (by some function that depends onlyon the size of the pattern graph).

Lemma 6. *Let $\mathbf{G} = (G, \emptyset, S_1, \ldots, S_l)$ and $\mathbf{D} = (D, B, Z_1, \ldots, Z_l)$ be two structures, where B is an independent set and $l \in \mathbb{N}$. Then $\mathbf{D} \leq_c \mathbf{G}$ if and only if there exists some $v \in V(G)$ such that $\mathbf{D} \leq_c \mathbf{G}_v^{(k)}$, where $k := \mathbf{diam}(D)$.*

Proof. Suppose first that $\mathbf{D} \leq_c \mathbf{G}$, i.e., there exists a function $\sigma : V(G) \to V(D)$ satisfying the contraction properties. Then $D \leq_c G$, and thus there exists a minor D' of G such that $D \simeq D'$. Moreover, notice that all vertices of $V(D) \setminus B$ are mapped to single vertices of G. Then, there exists a vertex $v \in V(D') \subseteq V(G)$ such that $D \setminus B \simeq D' \setminus (\sigma^{-1}(B)) \subseteq B_G^k(v)$. Consider now the structure $\mathbf{G}_v^{(k)} := (G_v, \{u_C \mid C \in \mathcal{C}(\tilde{G})\}, S'_1, \ldots, S'_l)$, where G_v is the k-contracted graph around the vertex v, $\tilde{G} = G \setminus B_G^k(v)$ and $S'_i = S_i \cap B_G^k(v)$, for every $i \in [l]$. Observe that $G_v = B_G^k(v) \uplus \{u_C \mid C \in \mathcal{C}(\tilde{G})\}$. Let then the function $\rho : V(G_v) \to V(D)$ be defined as follows.

- $\rho(u) = \sigma(u)$, for every $u \in V(B_G^k(v))$.
- $\rho(u_C) = v$, where $v \in B$ and there exists $u \in C$ such that $\sigma(u) = v$.

Note then that since $D' \setminus \sigma^{-1}(B) \subseteq B_G^k(v)$ it holds that $\rho^{-1}(D) = \sigma^{-1}(D)$. This implies that ρ satisfies the first three contraction properties. Moreover, since $\rho(u_C) \in B$, for every $C \in C(\tilde{G})$, then $\rho(\{u_C \mid C \in \mathcal{C}(\tilde{G})\}) \subseteq B$ and thus, ρ satisfies the fourth contraction property as well. Finally, as $S'_i \subseteq S_i \cap B_G^k(v)$, then ρ also satisfies the fifth contraction property. Therefore for the chosen $v \in V(G)$ we obtain that $\mathbf{D} \leq_c \mathbf{G}_v^{(k)}$.

Suppose now that there exists a vertex $v \in V(G)$ such that $\mathbf{D} \leq_c \mathbf{G}_v^{(k)}$. Then there exists a function $\sigma : V(G_v) \to V(D)$ satisfying the contraction properties. Notice that $\mathbf{G}_v^{(k)} \leq_c \mathbf{G}$ (where every vertex of $B_G^k(v)$ is mapped to itself and every vertex u_C is mapped to the connected component C). Thus, there exists a function $\rho : V(G) \to V(G_v)$ satisfying the contraction properties. It is easy to confirm that $\sigma \circ \rho : V(G) \to V(D)$ satisfies the contraction properties and therefore $\mathbf{D} \leq_c \mathbf{G}$. $\square$

We are now ready to prove the section's main result.

Proof of Lemma 5. First of all, we know that such a constant c exists from Lemma 3 and that in fact $c = \mathcal{O}(|E(\Delta)|)$. Since $\Delta \preceq \Gamma$, there exists an insertable set of non-edges $S \subseteq \overline{E(\Gamma)}$ and two plane graphs $\Gamma' = (V(\Gamma), E(\Gamma) \cup S)$ and Γ_0, such that $\Gamma_0 \subseteq \Gamma'$ and $\Delta \simeq_{tp} \Gamma_0$. Without loss of generality we may assume that all edges of S are also edges of Γ_0. Let then $\alpha : V(\Gamma_0)\cup F(\Gamma_0) \to V(\Delta)\cup F(\Delta)$ be a topological isomorphism between Γ_0 and Δ. For every edge $e = \{u, v\}$ of S let $e_\alpha = \{\alpha(u), \alpha(v)\}$. We define the sets $S_\alpha = \{e_\alpha \mid e \in S\}$, $S^\Delta_\alpha = S_\alpha \cap E(\Delta)$, and $S^\Gamma_\alpha = S_\alpha \setminus S^\Delta_\alpha$.

We first construct a graph $\Delta^+ \in \mathcal{F}_\Delta$. For this, we add a set of vertices and edges embedded inside some of the faces of Δ in such a way that edges intersect only at their common endpoints. In particular, for each face $f \in F(\Delta)$ with facial mapping $\mathbf{w}(f)$ do the following:

- For each edge $e = \{u, v\}$ that lies inside the region enclosed by $\alpha^{-1}(\mathbf{w}(f))$ in Γ and whose endpoints belong to Γ', add the edge $\{\alpha(u), \alpha(v)\}$ in the interior of f in Δ in such a way that (i) edges intersect only at their common endpoints and (ii) after we extend the mapping α so that it takes into account those edges of Γ that were added in Δ, the following must hold: for any connected component that was inside f and, after the addition of the edges, is in a face f', the preimages of the vertices of that connected component in Γ_0 are inside the region enclosed by $\alpha^{-1}(\mathbf{w}(f'))$.

- Consider the faces $f_1, f_2, \ldots, f_j$ that form the partition of f after the addition of the new edges. For every such face f_i let p_i be the region enclosed by $\alpha^{-1}(\mathbf{w}(f_i))$ in Γ'. Notice that since Δ^+ is connected, the boundary of $\alpha^{-1}(\mathbf{w}(f_i))$ is connected. For every $i \in [j]$ let $\mathcal{C}_{p_i}$ be the set of all connected components that lie entirely in the region enclosed by $\alpha^{-1}(\mathbf{w}(f_i))$ in Γ'. Let $\mathcal{C}^\emptyset_{p_i}$ denote the set of all connected components in $\mathcal{C}_{p_i}$ that do not have any neighbors in $B_{\Gamma'}(f_i)$. For every $C_w \in \mathcal{C}_{p_i}$, let S_w be its neighborhood in $B_\Gamma(f_i)$. Consider the Hasse diagram defined by the sets S_w and without loss of generality, let S_1, S_2, $\ldots$, S_q be its maximal elements. Let then $O_t = \{C_l \in \mathcal{C}_{p_i} \setminus \mathcal{C}^\emptyset_{p_i} \mid S_l \subseteq S_t\}$, $t \in [q]$. For every $t \in [q]$, add a vertex u_t in f_i and make it adjacent to the vertices in $\alpha(S_t)$ (notice that since the boundary is again connected there is a unique way to construct the cyclic neighborhood of u_t up to cyclic permutations). We call O_t the *origin* of u_t.

The resulting graph Δ^+ is, by definition, a member of $\mathcal{F}_\Delta$. See an example of such a construction in Figure 2.

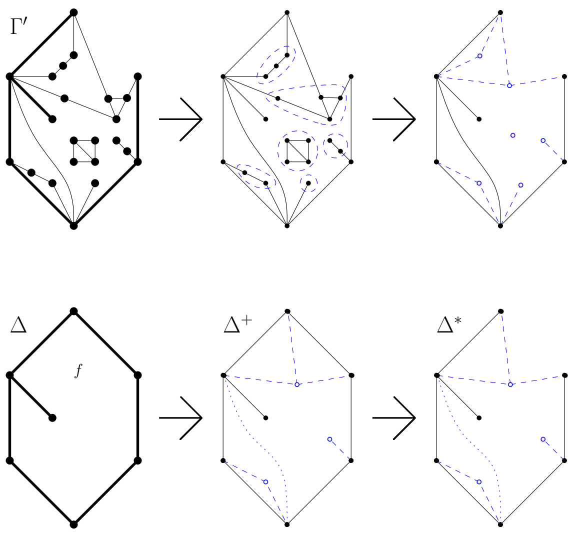

Figure 2: The construction of Δ^+. The new vertices are the vertices of $B = V(\Delta^*) \setminus V(\Delta)$.

To construct Δ^* from Δ^+, for every edge $\{u, v\} \in S$, we remove the edge $\{\alpha(u), \alpha(v)\}$ from Δ^+. Since $\{\alpha(u), \alpha(v)\} \in E(\Delta)$, it follows that

$$\Delta^* \in \mathsf{span}(\Delta^+, E(\Delta)).$$

We now define a function $g_0 : E(\Delta^*) \cup F(\Delta^*) \mapsto E(\Gamma) \cup F(\Gamma)$. Let $f \in F(\Delta^*)$ with facial mapping $\mathbf{w}(f)$. Observe that there is at least one face $f' \in F(\Gamma)$ with facial mapping $\mathbf{w}(f')$, such that for every facial walk $w = (u_1, \ldots, u_k) \in \mathbf{w}(f)$ there is a facial walk $w' \in \mathbf{w}(f')$ of length at least k and a subsequence $(v_1, \ldots, v_k)$ of w' (up to cyclic permutations) with the following properties: $v_i = \alpha(u_i)$ if $v_i \in \Delta$ and $v_i \in V(C)$, for some C in the origin of u_i, if $u_i \in V(\Delta^*) \setminus V(\Delta)$.

Notice that due to planarity the regions defined by those walks (unless the walks are trivial) are mutually nested. Of all such faces (if there are multiple), let f' be the one whose region contains all other regions. Then, $g_0(f) = f'$. We will call the connected component whose vertices belong to that walk the *outermost connected component*.

Recall that, by construction, the new vertices of $V(\Delta^*) \setminus V(\Delta)$ form an independent set. Thus, for each edge $e = \{u, v\} \in E(\Delta^*)$ at most one of its endpoints belongs in $V(\Delta^*) \setminus V(\Delta)$. If both endpoints u, v of e belong to $V(\Delta)$, then we define $g_0(e) = \{\alpha^{-1}(u), \alpha^{-1}(v)\} \in E(\Gamma)$. Otherwise exactly one of u and v, say v, belongs to $V(\Delta^*) \setminus V(\Delta)$. In this case, we define $g_0(e) = \{\alpha^{-1}(u), v'\} \in E(\Gamma)$, where v' is a neighbor of $\alpha^{-1}(u)$ in the outermost connected component in the origin of v.

Let now $R(\Gamma)$ be an arbitrary subdivided radial enhancement of Γ. In order to construct a subdivided radial enhancement $R(\Delta^*)$ of Δ recall that we first subdivide all edges of $R(\Delta^*)$ and then add a radial vertex u_f inside each face $f \in F(\Delta^*)$. For every f let $r_{g_0}(f)$ be the radial vertex of $R(\Gamma)$ that was added in $g_0(f)$. Consider the cyclic neighborhood of $r_{g_0}(f)$ in $R(\Gamma)$. Notice that it can be broken down in $s_1, s_2, \ldots, s_l$ segments where s_i is a facial walk w_i of $\mathbf{w}(g_0(f))$. Let w'_i be the subsequence of the walk that corresponds to a walk z_i in $\mathbf{w}(f)$. Add edges between the u_f and the vertices of the boundary of u_f in such a way that the cyclic neighborhood of u_f is $(z_1, z_2, \ldots, z_l)$. Notice that for every subdivision vertex x of $R(\Delta^*)$ that appears between u_i and u_{i+1} in the facial walk of w, there is a subdivision vertex v_x appearing between v_i and v_{i+1} in the walk w of $\mathbf{w}(f)$. We add an edge $\{u_f, v_x\}$ so that v_x appears between u_i and u_{i+1} in the cyclic neighborhood of u_f (this can be done in a unique way). We extend the mapping g_0 restricted to $E(\Delta^*)$ to the mapping g_1 by mapping every edge $\{u_f, u_i\}$ to the edge $\{r_{g_0}(f), v_i\}$. We also map the edges $\{u_f, x\}$ to the edges $\{r_{g_0(f)}, v_x\}$. Notice that g_1 can be extended to $F(R(\Delta^*))$ similarly to g_0. In the same fashion we extend g_1 to the function g_2 on the graphs $R^2(\Gamma)$ and $R^2(\Delta^*)$ and then to g_3 on the graphs $R^3(\Gamma)$ and $R^3(\Delta^*)$. Recall that

$$\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset) = (R^3(\Gamma), \emptyset, V(\Gamma), V_s^1(\Gamma), V_r^1(\Gamma), \ldots, V_s^3(\Gamma), V_r^3(\Gamma)),$$

and that

$$\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) =$$
$$= (R^3(\Delta^*), V(\Delta^*) \setminus V(\Delta), V(\Delta), V_s^1(\Delta^*), V_r^1(\Delta^*), \ldots, V_s^3(\Delta^*), V_r^3(\Delta^*)).$$

Let now $\sigma : V(R^3(\Gamma)) \to V(R^3(\Delta^*))$ such that:

$$\sigma(v) = \begin{cases} u & \text{if } v \in V(\Gamma), u \in V(\Delta), \text{ and } \alpha^{-1}(u) = v \in V(\Gamma) \\ z & \text{if } v \in V_s^i(\Gamma) \text{ and there exists } u \in V_s^i(\Delta^*) \text{ with } g^i(e) = e', i \in [3], \\ & \text{where } z \text{ (resp. } v\text{) is the subdivision vertex of the edge } e \text{ (resp. } e'\text{)} \\ w & \text{if } v \in V_r^i(\Gamma) \text{ and there exists } u \in V_r^i(\Delta^*) \text{ with } g^i(f) = f', i \in [3], \\ & \text{where } w \text{ (resp. } v\text{) is the radial vertex added in face } f \text{ (resp. } f'\text{)} \\ x & \text{where } x \in B \text{ such that the distance between } v \text{ and the vertices in} \\ & O_x \text{ in } R^3(\Gamma) \text{ is minimized} \end{cases}$$

It is quite straightforward to verify that σ satisfies the five required contraction properties and thus $\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) \leq_c \mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)$. Therefore, since these two structures satisfy the conditions of Lemma 6, we conclude that there exists some $v \in V(\Gamma)$ such that $\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) \leq_c \mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^{\mathbf{diam}(R^3(\Delta^*))}$. Notice now that $\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) \in \mathcal{H}_\Delta$ and that $\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^{\mathbf{diam}(R^3(\Delta^*))}$ is a minor of $\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^c \in \mathcal{G}_{\Gamma, R(\Gamma), c}$ and we have proven our claim. □

The next theorem is a direct consequence of Lemma 4 and Lemma 5.

Theorem 1. *Let Γ be a plane graph and Δ be a connected plane graph. It holds that $\Delta \preceq \Gamma$ if and only if for every $R(\Gamma) \in \mathcal{R}_\Gamma$ there exist two structures $\mathbf{G} \in \mathcal{G}_{\Gamma, R(\Gamma), c}$ and $\mathbf{D} \in \mathcal{H}_\Delta$, such that $\mathbf{D} \leq_c \mathbf{G}$, where c is a constant such that $\max_{\Delta^*}\{\mathbf{diam}(R^3(\Delta^*))\} \leq c$.*

Theorem 2. *There exists an algorithm that, given as input an n-edge plane graph Γ and a connected k-edge plane graph Δ, decides whether $\Delta \preceq \Gamma$ in $2^{\mathcal{O}(k \log k)} \cdot n^2$ steps.*

Proof. Let Γ and Δ be two plane graphs, where Δ is connected. From Theorem 1, we have $\Delta \preceq \Gamma$ if and only if for every $R(\Gamma) \in \mathcal{R}_\Gamma$ there exist two structures $\mathbf{G} \in \mathcal{G}_{\Gamma, R(\Gamma), c}$ and $\mathbf{D} \in \mathcal{H}_\Delta$, such that $\mathbf{D} \leq_c \mathbf{G}$, where c is a constant such that $\max\{\mathbf{diam}(R^3(\Delta^*)) \mid \Delta^* \in \mathcal{PG}_\Delta\} \leq c = \mathcal{O}(|E(\Delta)|)$. Thus, in the first step the algorithm fixes an arbitrary $R(\Gamma) \in \mathcal{R}_\Gamma$ and constructs the families $\mathcal{G}_{\Gamma, R(\Gamma), c}$ and $\mathcal{H}_\Delta$. By definition, $|\mathcal{G}_{\Gamma, R(\Gamma), c}| = n$ and from Lemma 3, $|\mathcal{H}_\Delta| = 2^{\mathcal{O}(k \cdot \log k)}$. Therefore the algorithm needs to check all possible pairs of structures which are $2^{\mathcal{O}(k \cdot \log k)} \cdot n$. To do that we can use the algorithm from Proposition 1, since from Lemma 2, the underlying graphs of all structures in $\mathcal{G}_{\Gamma, R(\Gamma), c}$ have treewidth $\mathcal{O}(k)$. The algorithm from Proposition 1 checks each pair in $2^{\mathcal{O}(k \cdot \log k)} \cdot n$ steps, therefore the whole algorithm outputs an answer in at most $2^{\mathcal{O}(k \cdot \log k)} \cdot n^2$ steps. For a more detailed description of the algorithm, see also Algorithm 1. □

Input : An encoding an n-edged plane graph Γ and a k-edged plane graph Δ.
Question: Is it true that $\Delta \preceq \Gamma$?

arbitrarily construct an $R(\Gamma) \in \mathcal{R}_\Gamma$;
foreach *facial extension* Δ^+ *of* Δ **do**
  foreach *pattern-guess* Δ^* *based on* Δ^+ **do**
    construct a member of $\mathcal{PG}_\Delta$;
    `//` $|\mathcal{PG}_\Delta| = 2^{\mathcal{O}(k \log k)}$ `by Lemma 3`
  end
end
construct $\mathcal{R}_{\Delta^*}$ *based on* $\mathcal{PG}_\Delta$;
construct the family $\mathcal{H}_\Delta$ *based on* $\mathcal{R}_{\Delta^*}$ *and* $\mathcal{PG}_\Delta$;
`//` $|\mathcal{H}_\Delta| = 2^{\mathcal{O}(k \log k)}$ `by Lemma 3`
set $c \geq \max\{\mathbf{diam}(R^3(\Delta^*)) \mid \Delta^* \in \mathcal{PG}_\Delta\}$;
`//` $c = \mathcal{O}(k)$ `by Lemma 3`
foreach $u \in V(\Gamma)$ **do** construct a member of the family $\mathcal{G}_{\Gamma,R(\Gamma),c}$;
foreach $\mathbf{G} \in \mathcal{G}_{\Gamma,R(\Gamma),c}$ **do** `//` n `steps`
  foreach $\mathbf{D} \in \mathcal{H}_\Delta$ **do** `//` $\mathcal{O}(k \log k)$ `steps`
    if $\mathbf{D} \leq_c \mathbf{G}$ **then return** *True*;
    `// in` $2^{\mathcal{O}(k \log c)} \cdot n$ `steps using the algorithm of Proposition 1 since the underlying graph of` $\mathbf{G}$ `has treewidth` $\leq 3(c+1)$ `by Lemma 2`
  end
end
return *False*;
`// the correctness of the algorithm is ensured by Theorem 1`

Algorithm 1: Algorithm of Theorem 2

4 An FPT algorithm for the PTMC problem

We need the following definitions and results before we are ready to prove the main result of this section.

Given a plane graph Γ and a non-negative integer k, we say that a graph Γ' is a k-face completion of Γ if it can be obtained from Γ in the following way; for every $f \in F(\Gamma)$ we add a set E_f of at most k edges to Γ such that the endpoints of the edges in E_f are vertices of Γ that belong to the boundary of f, all the edges E_f lie inside f, they do not intersect Γ in any points other than their endpoints, and finally they do not intersect each other.

Let r and q be integers such that $r \in \mathbb{N}_{\geq 3}, q \in \mathbb{N}_{\geq 1}$. A (r, q)-*cylinder*, denoted by $C_{r,q}$, is the Cartesian product of a cycle on r vertices and a path on q vertices. We will refer to r as the *length* and q as the *width* of $C_{r,q}$. Note here that $C_{r,q}$ is a 3-connected graph and thus, by Whitney's Theorem, it is uniquely embeddable (up to homeomorphism) in the sphere. Furthermore, $C_{r,q}$ has exactly two non-square faces f_1 and f_2 that are incident only with vertices of degree 3. We call one of the faces f_1 and f_2 the *interior* of $C_{r,q}$ and the other the *exterior* of $C_{r,q}$. We call the vertices incident to the interior (exterior) of $C_{r,q}$ *base* (*roof*) of $C_{r,q}$.

Let Γ be a plane graph. We give the definition of the graph $\Gamma_{r,q}$ for $r \in \mathbb{N}_{\geq 3}$ and $q \in \mathbb{N}_{\geq 3}$. Let $f_i \in F(\Gamma)$ and let $\Theta^i_1, \ldots, \Theta^i_{\rho_i}$ be the connected components of $B_\Gamma(f_i)$. For each Θ^i_j, we denote by $\sigma_{j,i}$ the length of a facial walk of Θ^i_j. We then add a copy C^i_j of $(\sigma_{j,i} \cdot r, q)$-cylinder in the embedding of Γ such that Θ^i_j is contained in the interior of C^i_j and all $\Theta^i_1, \ldots, \Theta^i_{j-1}, \ldots, \Theta^i_{j+1} \ldots, \Theta^i_{\rho_i}$ are contained in the exterior of C^i_j. Then we partition the base of C^i_j into $\sigma_{j,i}$ parts Q_l, $l \in \sigma_{j,i}$

each consisting of r consecutive base vertices. Let $(u^1_{j,i}, u^2_{j,i}, \ldots, u^{\sigma_{j,i}}_{j,i}, u^1_{j,i})$ be a facial walk of $\Theta_{j,i}$. We join by r edges the vertex $u^x_{j,i}$ to all the vertices of the set Q_l, $l \in \sigma_{j,i}$. We apply this enhancement for each connected component of the boundary of each face of Γ and we denote the resulting graph by $\widehat{\Gamma}_{r,q}$.

We call a face f_i of $\widehat{\Gamma}_{r,q}$ non-trivial if $B_{\widehat{\Gamma}_{r,q}}(f_i)$ has more than one connected components $\Theta^i_1, \ldots, \Theta^i_{\rho_i}$. Notice that if f_i is non-trivial, each Θ^i_j is the roof of some previously added cylinder. For each such cylinder, let J^i_j be a set of r consecutive vertices of its roof. We add inside f_i a copy C_{f_i} of $C_{\rho_i \cdot r, q}$ such that its base is a subset of f_i and let $\{I_1, \ldots, I_{\rho_i}\}$ be a partition of its roof in ρ_i parts, each consisting of r consecutive base vertices. For each $x \in \{1, \ldots, \rho_i\}$ we add r edges each connecting a vertex of J^i_j with some vertex of I_x in a way that the resulting embedding remains plane (there is a unique way for this to be done). We apply this enhancement for each non-trivial face of $\widehat{\Gamma}_{r,q}$ and the resulting graph is the graph $\Gamma_{r,q}$. Notice that $\Gamma_{r,q}$ is not uniquely defined as its definition depends on the choice of the sets J_x. From now on, we always consider an arbitrary choice for $\Gamma_{r,q}$ and we call $\Gamma_{r,q}$ the (r,q)-*cylindrical enhancement of* Γ. Finally, given a plane graph Γ and $r, q \in \mathbb{N}_{\geq 3}$. Let $V^0_{\Gamma,r,q} = V(\Gamma)$ and $V^n_{\Gamma,r,q} = V(\Gamma_{r,q}) \setminus V(\Gamma)$ and notice that $\deg_{\Gamma_{r,q}}(v) \leq 4$, for every $v \in V^n_{\Gamma,r,q}$. (For an example, see Figure 3.) Given a positive integer k, we denote by $\tilde{\Gamma}_k$ the graph $\Gamma_{2\cdot k, 8\cdot k}$.

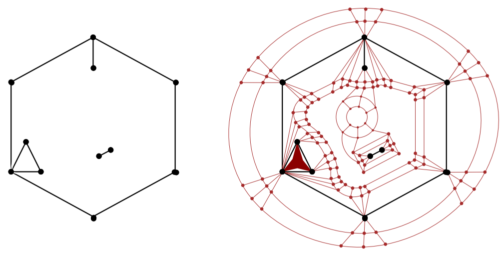

Figure 3: This figure depicts the construction of $\Gamma_{3,2}$ from Γ (Γ is the graph on the left).

Lemma 7 ([2]). *Let Γ be a plane graph and let f be a face of Γ. Let also M be a plane graph such that $M \subseteq \mathbf{cl}(f)$, $E(M) \cap E(B_\Gamma(f)) = \emptyset$ and $V(M) \subseteq V(B_\Gamma(f))$. Then there is a closed curve K in f meeting each edge of M twice.*

We define the *dual* of the plane graph Γ, denoted by $\Gamma' = (V', A')$ in the following way:

1. for every $f \in F(\Gamma)$, we add a vertex $v_f \in V'$ mapped to a point of f,

2. for every two faces $f, f' \in F(\Gamma)$ and every edge $e \in \mathbf{bd}(f) \cap \mathbf{bd}(f')$ we add an arc $e_{f,f'} \in A'$ joining v_f, $v_{f'}$ and meeting Γ only at a single point of e.

A planar graph G is *outerplanar* if there exists an embedding Γ in Σ and a face $f \in F(\Gamma)$ such that every vertex of Γ belongs to $\mathbf{bd}(f)$. We define the *weak dual* of such an embedding Γ of an outerplanar graph by removing from its dual the vertex that is contained in the interior of its unique face f that contains all vertices of Γ in its border.

Lemma 8. *Let Γ be a cycle with n vertices and k chords such that no two chords share a common endpoint. Let also Γ be embedded on Σ in such a way that if K is the curve defined by the vertices and the edges of Γ then all the chords of Γ are contained in its interior. Then Γ is an embedded topological minor of the $(n, k+1)$-cylinder C where the vertices of Γ are mapped on vertices of the roof of C and the paths joining the edges of the cycle contain only vertices and edges of the roof of C.*

Proof. We prove this lemma by induction on k. Notice that if Γ has at most one chord then the statement holds trivially.

Assume that the lemma holds for every cycle with $l < k$ chords and let Γ be a cycle with k chords. Notice that Γ is an outerplanar graph and consider its weak-dual, which by definition is a tree, say T. Let e_i, $i \in [k]$ be the chords that bound the faces of Γ that correspond to leaves of T. We call these chords simplicial. From the induction hypothesis $\Gamma \setminus \{e_i \mid i \in [k]\}$ is a topological minor of the $(n, k-i+1)$-cylinder, say C', and the paths joining the edges of Γ contain only vertices and edges of the roof of C'. Let C'' be the graph obtained from the C' by subdiving each edge of C' that has exactly one endpoint in its roof exactly once and add edges between the subdivided edges so as to obtain the $(n, k-i+2)$-cylinder, say C. Notice then that $\Gamma \setminus \{e_i \mid i \in [k]\}$ is an embedded topological minor of C such that the paths joining the edges of Γ contain only vertices and edges of the roof of C. Finally, since no two chords of Γ share a common endpoint observe that we may find paths joining the simplicial edges $\{e_i \mid i \in [k]\}$ by using the newly added vertices and edges of C. By the construction, it is easy to verify that the paths joining the edges of Γ in C form indeed the model of an embedded topological minor of Γ. $\square$

Observation 4. *Let $r \in \mathbb{N}_{\geq 3}$ and $q \in \mathbb{N}_{\geq 2}$. For every plane graph Γ, $|V(\Gamma_{r,q})| = \mathcal{O}(|V(\Gamma)|)$ and $|E(\Gamma_{r,q})| = \mathcal{O}(|E(\Gamma)|)$.*

Let G be a graph. We say that two paths in G are *disjoint* if they don't share common internal vertices. Given a collection $\mathcal{P}$ of pairwise disjoint paths of G, we define $L(\mathcal{P}) = \{\{x, y\} \mid x \text{ and } y \text{ are the endpoints of a path in } \mathcal{P}\}$.

Given a plane graph Γ and an open set Λ of Σ, we define $\Gamma\langle\Lambda\rangle$ as the graph whose edge set consists of the edges of Γ that are subsets of Λ and whose vertex set consists of their endpoints.

Proposition 2. *Let Γ be a plane graph, $k \in \mathbb{N}_{\geq 1}$ and let Γ^+ be a k-face completion of Γ. For every face $f \in F(\Gamma)$, there is a collection $\mathcal{P}$ of k disjoint paths in the graph $\tilde{\Gamma}_k\langle f\rangle$ such that $L(\mathcal{P}) = E(\Gamma^+\langle f\rangle)$.*

Proof. Let $f_i \in F(\Gamma)$ be a face of Γ. By the assumption, there are at most k edges of Γ^+ contained in f_i. Let $\{s_q, t_q\}$, $q \in \{1, \ldots, k\}$ be those k edges. From now on we will call them completion edges. Let also $\Theta^i_1, \Theta^i_2, \ldots, \Theta^i_{\rho_i}$ be the connected components of $B_\Gamma(f_i)$. Notice that for every completion edge e either there exists a Θ^i_j such that both endpoints of e are vertices of Θ^i_j or there exist disjoint Θ^i_j and $\Theta^i_{j'}$ where each of them contains exactly one of the endpoints of e. For every edge e for which there exists an Θ^i_j that contains both endpoints of e, we say that e is *internal* to Θ^i_j. For every edge e for which there exist disjoint Θ^i_j and $\Theta^i_{j'}$ where each of them contains exactly one of the endpoints of e we say that e is an *outgoing* edge of Θ^i_j and $\Theta^i_{j'}$.

We first consider the case where $\rho_i \geq 2$. From Lemma 7, there exists a closed curve K_{f_i} in f_i meeting each completion edge twice such that $\mathbf{int}(K_{f_i})$ contains only points of f_i. Moreover, notice that for every connected component Θ^i_j there exists a closed curve $K^i_j \subseteq \Sigma \setminus \mathbf{cl}(K_{f_i})$ in f such that (1) $\Theta^i_j \subseteq \mathbf{int}(K^i_j)$, (2) K^i_j meets each internal edge of Θ^i_j exactly twice, (3) K^i_j meets each outgoing edge of Θ^i_j exactly once and (4) for $j \neq j'$, $K^i_j \cap K^i_{j'} = \emptyset$. Finally, observe that we may choose the curves K^i_j, $j \in [\rho_i]$, and K_{f_i} in such a way that $K_{f_i} \subseteq \Sigma \setminus \mathbf{cl}(K^i_j)$, $j \in [\rho_i]$.

At every point in Σ where a completion edge meets a curve K_{f_i} or K^i_j, $j \in [\rho_i]$ we add a new vertex. For every $j \in [\rho_i]$, we denote the set of new vertices on K^i_j by V^i_j. We also denote by $V^i_{f_i}$ the vertices on K_{f_i}. Note here that by definition $|V^i_{f_i}|, |V^i_j| \leq 2 \cdot k$. Notice that every vertex $v \in V^i_{f_i}$ has exactly one neighbor in $\bigcup_{j \in [\rho_i]} V^i_j$. That is, for every vertex

$v \in V^i_{f_i}$ there exists exactly one $j \in [\rho_i]$ such that $N(v) \cap V^i_j \neq \emptyset$. We partition $V^i_{f_i}$ to the sets $\tilde{V}^i_j = \{v \in V^i_{f_i} \mid N(v) \cap V^i_j \neq \emptyset\}$. Finally observe that, due to the planarity of the embedding, for every $j \in [\rho_i]$ the vertices of $\tilde{V}^i_j$ appear consecutively in K_{f_i}.

Recall that for every $j \in [\rho_i]$, Θ^i_j defines a closed curve in Σ and let $A_j = A[\Theta^i_j, K^i_j]$ denote the annulus between the curve defined by Θ^i_j and K^i_j. Without loss of generality we may assume that the cylinder C^i_j of $\Gamma_{2\cdot k,8\cdot k}$ is embedded inside A_j in such a way that V^i_j are consecutive vertices of the roof of C^i_j and that $V^i_j \subseteq J^i_j$. It is then easy to see that for every edge inside the annulus A_j (which is part of a completion edge) we may find a path in C^i_j joining its endpoints in Γ_k.

Let now $\mathbf{int}(K_{f_i})$ be the open disk of $\Sigma \setminus K_{f_i}$ that contains only points of the completion edges. Without loss of generality we may also assume that the cylinder C_{f_i} is embedded inside $\mathbf{int}(K_{f_i})$ in such a way that $\tilde{V}^i_j \subseteq I_j$, for every $j \in [\rho_i]$. Observe that the curve K_{f_i} together with the edges in its interior forms a cycle Q on $2 \cdot k$ vertices with k chords where no two chords share a common endpoint. From Lemma 8, Q is an embedded topological minor of the $(2 \cdot k, k + 1)$-cylinder C such that all vertices of Q are mapped on vertices of the roof of C and the paths joining the edges of the cycle contain only vertices and edges of the roof of C. Thus, by appropriately subdividing C we may see that Q is also a topological minor of C_{f_i}. Therefore, for the edges in $\mathbf{int}(K_{f_i})$ there exist vertex disjoint paths in C_{f_i} joining their endpoints. This concludes the proof that there exists a collection $\mathcal{P}$ of k disjoint paths in the graph $\tilde{\Gamma}_k\langle f_i\rangle$ such that $L(\mathcal{P}) = E(\Gamma^+\langle f_i\rangle)$.

In the special case where $\rho_i = 1$, let us consider only the closed curve K_{f_i} in f_i that meets each completion edge twice and add a vertex at every point where a completion edge meets a curve K_{f_i}. Then as above the curve K_{f_i} together with the edges in its interior forms a cycle Q on $2 \cdot k$ vertices with k chords where no two chords share a common endpoint. Again, from Lemma 8, Q is an embedded topological minor of the $(2 \cdot k, k + 1)$-cylinder C such that all vertices of Q are mapped on vertices of the roof of C and the paths joining the edges of the cycle contain only vertices and edges of the roof of C. Thus, by appropriately subdividing C we may see that Q is also a topological minor of C^i_1. Therefore, for the edges in $\mathbf{int}(K_{f_i})$ there exist vertex disjoint paths in C^i_1 joining their endpoints. This concludes the proof that there exists a collection $\mathcal{P}$ of k disjoint paths in the graph $\tilde{\Gamma}_k\langle f_i\rangle$ such that $L(\mathcal{P}) = E(\Gamma^+\langle f_i\rangle)$. $\square$

From the proof of [2, Theorem 2] we obtain the following lemma.

Lemma 9. *Let k be a positive integer. Let Γ be a plane graph and $\{(s_i, t_i) \mid i \in [k]\}$ be k pairs of terminals of Γ such that the graph with vertex set $\bigcup_{i\in[k]}\{s_i, t_i\}$ and edge set $\{\{s_i, t_i\} \mid i \in [k]\}$ is connected. Let also f be a face of Γ and Γ^+ be a face completion of f. If Γ^+ contains k internally vertex disjoint paths joining the terminals (s_i, t_i), then there exists a face completion Γ' of f with at most k^{2^k} edges.*

We are now ready to prove one of the main results of this section.

Theorem 3. *Let Γ and Δ be plane graphs where Δ is connected and $k = |E(\Delta)|^{2^{|E(\Delta)|}}$. There exists a k-face completion Γ^+ of Γ such that $\Delta \leq_{etm} \Gamma^+$ if and only if $\Delta \leq^S_{etm} \tilde{\Gamma}_k$ where $S = V(\tilde{\Gamma}_k) \setminus V(\Gamma) = V^n_{\Gamma,2\cdot k,8\cdot k}$.*

Proof. Suppose that $\Delta \leq_{etm} \tilde{\Gamma}_k$, no vertex of Δ is mapped to a vertex of $V(\tilde{\Gamma}_k) \setminus V(\Gamma)$, and that $\mathcal{P}$ is the collection of disjoint paths in $\tilde{\Gamma}_k$, corresponding to edges of Δ. Let $\mathcal{P}'$ be family of the maximal (according to the number of vertices) subpaths of the paths in $\mathcal{P}$ that consist only of edges of $E(\tilde{\Gamma}_k) \setminus E(\Gamma)$). Let Γ^+ be the graph obtained from $\tilde{\Gamma}_k$ by removing all vertices of $(\tilde{\Gamma}_k \setminus \Gamma)$ that do not belong to any path of $\mathcal{P}'$ and contracting all paths in $\mathcal{P}'$ to a single edge. Then by construction, Γ^+ is a completion of Γ such that $\Delta \leq_{etm} \Gamma^+$.

We now show that if there exists a a completion Γ^+ of Γ such that $\Delta \leq_{etm} \Gamma^+$ then $\Delta \leq_{etm} \tilde{\Gamma}_k$ and no vertex of Δ is mapped to a vertex of $V(\tilde{\Gamma}_k) \setminus V(\Gamma)$. Let Γ^+ be a completion of Γ such that $\Delta \leq_{etm} \Gamma^+$ where $|E(\Gamma^+) \setminus E(\Gamma)|$ is minimal. From Lemma 9 we may assume that in each face $f \in F(\Gamma)$ at most k edges have been added in Γ^+. Suppose not and let $f \in F(\Gamma)$ be a face of Γ where more than k edges have been added in Γ^+. Let also $\{\{s_i, t_i\} \mid i \in |E(\Delta)|\}$ be the pairs of vertices of Γ corresponding to the endpoints of the edges of Δ. Clearly, in Γ^+ there exist internally vertex disjoint paths joining the pairs $\{s_i, t_i\}$. Let $\hat{\Gamma}$ be the graph obtained from Γ^+ after we remove the completion edges that have been added to the face f of Γ in it. As Γ^+ and $\hat{\Gamma}$ only differ in the edges of f and f is a face of $\hat{\Gamma}$ there clearly exists a completion of the face f of $\hat{\Gamma}$ (that is, Γ^+) such that there exist $|E(\Delta)|$ internally vertex disjoint paths joining the pairs $\{s_i, t_i\}$. From Lemma 9, there exists a face completion Γ' of f in $\hat{\Gamma}$ with at most k edges. However, Γ' is also a completion of Γ that contains $|E(\Delta)|$ internally vertex disjoint paths joining the pairs $\{s_i, t_i\}$. Moreover, in every face of Γ except f the same amount of edges as in Γ^+ has been added and strictly less edges have been added in f, a contradiction to the minimality of Γ^+.

Proposition 2 yields that for every face $f \in F(\Gamma)$ there exist a collection $\mathcal{P}$ of k disjoint paths in $\tilde{\Gamma}_k\langle f \rangle$ such that $L(\mathcal{P}) = E(\Gamma^+\langle f \rangle)$. Therefore, we conclude that $\Delta \leq_{etm} \tilde{\Gamma}_k$ and by construction no vertex of Δ is mapped to a vertex of $V(\tilde{\Gamma}_k) \setminus V(\Gamma)$. □

Proposition 3 ([11, 10])**.** *There exists an algorithm A_1 that given a planar graph G and a non-negative integer q outputs either a tree decomposition of G of width at most $18q$ or a subdivided wall W of G of height q and a tree decomposition $\mathcal{D}$ of the compass K_W of W of width at most $18q$ in time $O_q(|E(G)|)$.*

Proposition 4 ([11])**.** *There exists a function $f : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 0}$ and an algorithm A_2 that, given a planar graph Γ, an k-vertex planar graph Δ, and a subdivided wall W as a subgraph of Γ of height $f(k)$ whose compass has treewidth at most ℓ, outputs an edge $e = \{u, v\}$ of Γ where $\deg_\Gamma(v), \deg_\Gamma(u) \geq 2$ such that $\Delta \leq_{etm} \Gamma$ if and only if $\Delta \leq_{etm} \Gamma \setminus \{e\}$. This algorithm runs in $O_{k,\ell}(|E(\Gamma)|)$ steps.*

Lemma 10. *Let Γ and Δ be two plane graphs and $V \subseteq V(\Gamma)$, where $l = \max\{\deg_\Gamma(v) \mid v \in V\}$. There exist two plane graphs Γ' and Δ' such that*

- $\deg_{\Gamma'}(v) = 1$, *for every $v \in V(\Gamma') \setminus V(\Gamma)$, every edge in $E(\Gamma') \setminus E(\Gamma)$ has one endpoint in $\Gamma' \setminus \Gamma$,*
- $|E(\Gamma')| \leq (2l + 3)|E(\Gamma)|$, $|E(\Delta')| \leq (2l + 3)|E(\Delta)|$, $\mathbf{tw}(\Gamma') = \mathbf{tw}(\Gamma)$, *and*
- $\Delta \leq_{etm}^{V} \Gamma$ *if and only if* $\Delta' \leq_{etm}^{\emptyset} \Gamma'$.

Moreover if $\tilde{\Gamma}$ is a spanning subgraph of Γ' such that $\Delta' \leq_{etm} \tilde{\Gamma}$ then $\Delta \leq_{etm}^{V} \Gamma \setminus E$, where $E = E(\Gamma') \setminus E(\tilde{\Gamma})$.

Proof. For every vertex $v \in V(\Gamma) \setminus V$ let $(v_1, v_2, \ldots, v_k)$ be the cyclic neighborhood of v. We add $(l + 1)k$ new pendant neighbors to v in such a way that in its new cyclic neighborhood exactly $(l + 1)$ vertices appear between v_i and v_{i+1}, $i \in [k]$, where $v_{k+1} = v_1$. Let Γ' denote the new graph.

For every vertex $u \in V(\Delta)$ let $(u_1, u_2, \ldots, u_p)$ be the cyclic ordering of the neighborhood of u. We add $(l + 1)p$ new pendant neighbors to u in such a way that in the new cyclic ordering exactly $(l + 1)$ vertices appear between u_i and u_{i+1}, $i \in [p]$, where $u_{p+1} = u_1$. Let Δ' denote the new graph.

Let f be an embedded topological minor model of Δ in Γ where every vertex of Δ is mapped to a vertex of $V(\Gamma) \setminus V$. Then by using the newly added pendant vertices in Γ' we may extend it to an embedded topological minor model of Δ' in Γ'.

Conversely, let f be an embedded topological minor model of Δ' in Γ'. Notice that all vertices of $V(\Delta)$ in Δ' have degree greater than l. Observe also that by construction all vertices of $V(\Gamma')$ that are not vertices of $V(\Gamma) \setminus V$ are the vertices of V and the newly added pendant vertices and thus, have degree at most l. This implies that no vertex of Δ can be mapped to a vertex of V in Γ'. It follows that if $e \in E(\Delta)$ then the path joining its endpoints in the model of Δ' in Γ' does not contain any of the newly added pendant edges. Therefore, the restriction of f to vertices of Δ and paths that model edges of Δ leads to an embedded topological minor model of Δ in Γ where no vertex of Δ has been mapped to a vertex of V.

Let now $\tilde{\Gamma}$ be a spanning subgraph of Γ', where $E = E(\Gamma') \setminus E(\tilde{\Gamma})$, such that Δ' is an embedded topological minor of $\tilde{\Gamma}$. Since the degree of every vertex in $\tilde{\Gamma}$ does not increase after removing edges, as above we obtain that if f is an embedded topological minor model of Δ in $\tilde{\Gamma}$ no vertex of Δ has been mapped to a vertex of V. Therefore, again, the restriction of f to vertices of Δ and paths that model edges of Δ leads to an embedded topological minor model of Δ in $\Gamma \setminus E$ where no vertex of Δ has been mapped to a vertex of V.

Observe that by construction $|E(\Gamma')| \leq (2l+3)|E(\Gamma)|$ and $|E(\Delta')| \leq (2l+3)|E(\Delta)|$. Finally, the fact that $\mathbf{tw}(\Gamma') = \mathbf{tw}(\Gamma)$ follows from the folklore observation that the addition of pendant vertices on a graph does not increase its treewidth. $\square$

Proposition 5. *Let c be a fixed positive integer. Let Δ be a k-edge plane graph and Γ be a plane graph with $\mathbf{tw}(\Gamma) > 18f((2c+3)\cdot k)$, where $f : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 0}$ is the function of Proposition 4 and $V \subseteq V(\Gamma)$ where $\deg_\Gamma(z) \leq c$, for every $z \in V$. Then, there exists an algorithm A that, given Γ and Δ, outputs an edge $e = \{u, v\} \in E(\Gamma)$ such that $\deg_\Gamma(v), \deg_\Gamma(u) \geq 2$ and $\Delta \leq^V_{etm} \Gamma$ if and only if $\Delta \setminus \{e\} \leq^V_{etm} \Gamma$. This algorithm runs in $O_k(|E(\Gamma)|)$ steps.*

Proof. Observe first that for every edge $e \in E(\Gamma)$, if $\Delta \leq^V_{etm} \Gamma \setminus \{e\}$ then clearly $\Delta \leq^V_{etm} \Gamma$. Let Γ' and Δ' be the graphs of Lemma 10. Run algorithm A_1 of Proposition 3 with input Γ' and $f((2c+3)\cdot k)$. As $\mathbf{tw}(\Gamma) > 18f((2c+3)\cdot k)$, A_1 outputs a subdivided wall W of height $f((2c+3)\cdot k)$ that is a subgraph of Γ' and whose compass K_W has treewidth at most $18f((2c+3)\cdot k)$.
Run algorithm A_2 of Proposition 4 with input Γ', Δ' and W. The output of A_2 is an edge $e = \{u, v\} \in E(\Gamma')$ such that $\Delta' \leq_{etm} \Gamma'$ if and only if $\Delta' \leq \Gamma' \setminus \{e\}$. Recall that all edges of $E(\Gamma') \setminus E(\Gamma)$ have one endpoint of degree 1. Thus, e is an edge of Γ. We will prove that $\Delta \leq^V_{etm} \Gamma$ if and only if $\Delta \leq^V_{etm} \Gamma \setminus \{e\}$. Observe first that for every edge $e' \in E(\Gamma)$, if $\Delta \leq^V_{etm} \Gamma \setminus \{e'\}$ then clearly $\Delta \leq^V_{etm} \Gamma$. Thus, let us assume that $\Delta \leq^V_{etm} \Gamma$. Then from Lemma 10, $\Delta' \leq_{etm} \Gamma'$. Therefore, from the choice of e, $\Delta' \leq_{etm} \Gamma' \setminus \{e\}$. Since $\Gamma' \setminus \{e\}$ is a spanning subgraph of Γ', again from Lemma 10, $\Delta \leq^V_{etm} \Gamma \setminus \{e\}$. Since $|E(\Gamma')| \leq (2c+3)\cdot |E(\Gamma)|$, it is clear that the overall algorithm runs in $O_k(|E(\Gamma)|)$ steps. $\square$

The proof of the following theorem follows by repetitive applications of the algorithm of Proposition 5.

Theorem 4. *There exists an algorithm that given two plane graphs Γ and Δ and a set $V \subseteq V(\Gamma)$ with $\deg_\Gamma(z) \leq c$, for every $z \in V$ outputs a graph Γ', with $\Gamma' \subseteq_{sp} \Gamma$ and $\mathbf{tw}(\Gamma') = \mathcal{O}(f(|E(\Delta)|))$, for some computable function f such that $\Delta \leq^V_{etm} \Gamma$ if and only if $\Delta \leq^V_{etm} \Gamma'$. This algorithm runs in $\mathcal{O}_{|E(\Delta)|}(|E(\Gamma)|)$ steps.*

Let Γ be a connected plane graph and $Z \subseteq V(\Gamma)$, we define the following pair of vertex and edge structures:

$$\mathbf{G}_{\Gamma,Z} := (\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), Z), \mathbf{e}(\Gamma, R(\Gamma))).$$

Given two connected plane graphs Δ and Γ and $Z \subseteq V(\Gamma)$ we say that $\mathbf{G}_{\Delta,\emptyset}$ is a *restricted topological minor* of $\mathbf{G}_{\Gamma,Z}$, denoted by $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$, if and only if there exist two functions $f_1 : V(R^3(\Delta)) \to 2^{V(R^3(\Gamma))}$ and $f_2 : E(R^3(\Delta)) \to 2^{E(R^3(\Gamma))}$ satisfying the following:

1. for every $x \in V(\Delta)$, $f_1(x) \in V(\Gamma) \setminus Z$ and $|f_1(x)| = 1$,
2. for every $x \in \cup_{i\in[3]} V(R_s^i(\Delta))$, $f_1(x) \notin \cup_{i\in[3]}(V(R_r^i(\Gamma)))$ and $|f_1(x)| = 1$,
3. for every $x, y \in \cup_{i\in[3]} V(R_r^i(\Delta))$ is connected and $f_1(x) \cap f_1(y) = \emptyset$,
4. for every $xy \in E_s^3(\Delta)$, $G[f_2(xy)]$ is a path between $f_1(x)$ and $f_1(y)$ and $f_2(xy) \subseteq E_s^3(\Gamma)$, and
5. for every $xy \in E_r^3(\Delta)$, $G[f_2(xy)]$ is a path between some vertex of $f_1(x)$ and some vertex of $f_1(y)$.

We will use the following three observations:

Observation 5. *Let Δ, Γ and Θ be three plane graphs, Θ_s be a subdivision of Θ, and $Z \subseteq V(\Gamma)$. If Δ is topologically isomorphic to Θ and $\Theta_s \leq_{es}^Z \Gamma$, then $\Delta \leq_{etm}^Z \Gamma$.*

Observation 6. *Let Θ be a connected plane graph and Γ be a plane graph. If $\Theta \leq_{es}^Z \Gamma$, then $\mathbf{G}_{\Theta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$.*

Observation 7. *If a connected plane graph Γ is a subdivision of a connected plane graph Θ, then $\mathbf{G}_{\Theta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,\emptyset}$.*

Theorem 5. *Let Γ, Δ be two connected plane graphs and $Z \subseteq V(\Gamma)$. Then $\Delta \leq_{etm}^Z \Gamma$ if and only if $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$.*

Proof. Let us make clear what kind of information about Γ and Δ is encoded in the structures $\mathbf{G}_{\Gamma,Z}$ and $\mathbf{G}_{\Delta,\emptyset}$.

We first prove that $\Delta \leq_{etm}^Z \Gamma$ implies that $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$. Suppose that $\Delta \leq_{etm}^Z \Gamma$. Then, there exist functions $\rho_1 : V(\Delta) \to V(\Gamma)$ and $\rho_2 : E(\Delta) \to \mathcal{P}(\Gamma)$ that satisfy conditions (1)-(3) of the definition of embedded topological minors and let Θ be the union of all paths in $\rho_2(E(\Delta))$. It is clear that Θ is connected and is a subdivision of $\Gamma\langle\rho_2\rangle$, that does not contain any vertex in Z, thus we get, from Lemma 7, that $\mathbf{G}_{\Gamma\langle\rho_2\rangle,\emptyset} \leq_{rtm} \mathbf{G}_{\Theta,Z}$. We also have that $\Theta \leq_{es}^Z \Gamma$ (we just delete every vertex of Γ that does not belong to any path of $\rho_2(E(\Delta))$) and thus from Lemma 6 that $\mathbf{G}_{\Gamma\langle\rho_2\rangle,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$. Finally, as Δ is topologically isomorphic to $\Gamma\langle\rho_2\rangle$, we get that $\mathbf{G}_{\Delta\langle\rho_2\rangle,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$, which is what we wanted to prove.

Now we prove that $\mathbf{G}_{\Delta\langle\rho_2\rangle,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$ implies that $\Delta \leq_{etm}^Z \Gamma$.
For any connected plane graph Λ, let $V^r(\Lambda) = V_r^1(\Lambda) \cup V_r^2(\Lambda) \cup V_r^3(\Lambda)$ and $V^s(\Lambda) = V_s^1(\Lambda) \cup V_s^2(\Lambda) \cup V_s^3(\Lambda)$. Suppose that $\mathbf{G}_{\Delta\langle\rho_2\rangle,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$. Then, there exist two functions $f_1 : V(H) \to 2^{V(G)}$ and $f_2 : E(H) \to 2^{E(G)}$ that satisfy conditions (1)-(5) of the definition of restricted topological minors, where $H = R^3(\Delta)$ and $G = R^3(\Gamma)$. Let Ξ be the connected graph obtained by the union of all paths in $f_2(E(H))$. Observe that by deleting from H all vertices in $V^r(\Delta)$ and by dissolving all vertices in $V^s(\Delta)$ we obtain the initial plane graph Δ. Let G' be the connected plane graph obtained by deleting from $G\langle f_2\rangle$ all vertices in $V^r(\Gamma)$ and G^* the connected plane graph obtained by dissolving all vertices of $V^s(\Gamma)$ in G'. As H and $G\langle f_2\rangle$ are 3-connected

and thus uniquely embedded, we finally obtain that Δ and G^* are topologically isomorphic. Moreover, G' is a subdivision of G^* and $G' \leq_{es}^{Z} \Gamma$, as it is obtained from Γ by deleting every vertex that does not belong to any path of $f_2(E(\Delta))$. Thus, from Lemma 5, we get that $\Delta \leq_{etm}^{Z} \Gamma$, which is what we want. $\square$

Our algorithm for PTMC. Let Γ and Δ be two plane graphs, where Δ is connected. From Therorem 3 we construct a cylindrical enhancement $\tilde{\Gamma}_k$ of Γ, where the vertices of the set $S = V^{n}_{\Gamma,2\cdot k,8\cdot k}$ have degree bounded by a constant such that Δ, Γ are a yes instance if and only if $\Delta \leq_{etm}^{S} \tilde{\Gamma}_k$. Then, the algorithm of Theorem 3 with inputs $\tilde{\Gamma}_k, \Delta, S$ outputs a graph Γ' with $\Gamma' \subseteq_{sp} \Gamma$ and $\mathbf{tw}(\Gamma') = \mathcal{O}(f(|E(\Delta)|))$. Moreover, Theorem 5 translates Γ', Δ, and S to two structures $\mathbf{G}_{\Delta,\emptyset}$ and $\mathbf{G}_{\Gamma,S}$, for which $\Delta \leq_{etm}^{S} \Gamma$ if and only if $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,S}$. Notice that the relation $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,S}$ can be expressed in Monadic Second Order Logic. Finally, by observing that $\mathbf{tw}(R^3(\Gamma)) = \mathcal{O}(f(|E(\Delta)|))$ we can employ Courcelle's Theorem [6] to obtain an $f(|E(\Delta)|) \cdot m^2$ time algorithm, for some computable function f.

5 Extensions

Our approach for the PSC problem can also solve the Plane Induced Subgraph Completion problem, with the same running time, where instead of an embedded subgraph we ask for an embedded *induced subgraph*. The only modification would be at the definition of a facial extension of Δ where we would additionally require that every connected graph Δ^+ contains Δ as an *induced* subgraph.

In the PTMC problem the connectivity of Δ is only required in the proof of Proposition 3 (that has been moved to the appendix). We would like to remark here that if we disregard the embedding of Δ then the Proposition holds for disconnected graphs as well. In this case by modifying the algorithm for PTMC we may obtain an FPT algorithm that given a plane graph Γ and a *planar* graph D decides whether there exists a face completion of Γ, say Γ^+, such that D is a rooted topological minor of Γ. That is, each vertex of D is mapped to a specified vertex of Γ. Notice that this approach also permits us to solve the Planar Disjoint Paths Completion problem where we allow edge additions inside all faces of Γ (in contrast to [2] where edge additions are allowed only inside a specified face of Γ).

Finally, with the same cylindrical enhancement that we apply for PTMC and the extra restriction that the sets of vertices of the enhanced graph that are contracted to a vertex of the pattern graph Δ contain only vertices of the initial graph we can solve the Plane Minor Completion problem. In these last two cases, however, only the existence of an FPT algorithm is verified (since both would be derived by Courcelle's Theorem).

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