The expected number of perfect matchings in cubic planar graphs

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Abstract: A well-known conjecture by Lovász and Plummer from the 1970s asserting that a bridgeless cubic graph has exponentially many perfect matchings was solved in the affirmative by Esperet et al. (Adv. Math. 2011). On the other hand, Chudnovsky and Seymour (Combinatorica 2012) proved the conjecture for the special case of cubic *planar* graphs. In our work we consider random bridgeless cubic planar graphs with the uniform distribution on graphs with n vertices. Under this model we show that the expected number of perfect matchings in *labeled* bridgeless cubic planar graphs is asymptotically $c\gamma^n$, where $c > 0$ and $\gamma \sim 1.14196$ is an explicit algebraic number. We also compute the expected number of perfect matchings in (non necessarily bridgeless) cubic planar graphs and provide lower bounds for *unlabeled* graphs. Our starting point is a correspondence between counting perfect matchings in rooted cubic planar maps and the partition function of the Ising model in rooted triangulations.

1 Introduction

In the 1970s Lovász and Plummer conjectured that a bridgeless cubic graph has exponentially many perfect matchings. The conjecture was solved in the affirmative by Esperet, Kardoš, King, Král and Norine [**5**], and independently for cubic planar graphs by Chudnovsky and Seymour [**4**]. The lower bound from [**5**] is $2^{n/3656} \approx 1.0002^n$. It is natural to expect that a typical bridgeless cubic graph has more perfect matchings than those guaranteed by this lower bound.

Our main result gives estimates on the expected number of perfect matchings, both for labeled and unlabeled cubic planar graphs. The model we consider is the uniform distribution on graphs with n vertices.

Theorem 1 *Let X_n be the number of perfect matchings in a random (with the uniform distribution) labeled bridgeless cubic planar graph with $2n$ vertices. Then*

$$\mathbf{E}(X_n) \sim b\gamma^n,$$

where $b > 0$ is a constant and $\gamma \approx 1.14196$ is an explicit algebraic number. If X_n^u is the same random variable defined on unlabeled bridgeless cubic planar graphs, then

$$\mathbf{E}(X_n^u) \geq 1.119^n.$$

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We obtain a similar result for general, non necessarily bridgeless, cubic planar graphs.

Theorem 2 *Let Y_n be the number of perfect matchings in a random (with the uniform distribution) labeled cubic planar graph with $2n$ vertices. Then*

$$\mathbf{E}(Y_n) \sim c\delta^n,$$

where $c > 0$ is a constant and $\delta \approx 1.14157$ is an explicit algebraic number. If Y_n^u is the same random variable defined on unlabeled cubic planar graphs, then

$$\mathbf{E}(Y_n^u) \geq 1.109^n.$$

2 Preliminaries

A map is a planar multigraph with a specific embedding in the plane. All maps considered in this paper are rooted, that is, an edge is marked and given a direction. A map is *simple* if it has no loops or multiple edges. It is 2-connected if it has no loops or cut vertices, and 3-connected if it has no 2-cuts or multiple edges. A map is cubic if it is 3-regular, and it is a triangulation if every face has degree 3. By duality, cubic maps are in bijection with triangulations. And since duality preserves 2- and 3-connectivity, k-connected cubic maps are in bijection with k-connected triangulations, for $k = 2, 3$. Notice that a general triangulation can have loops and multiple edges, and that a simple triangulation is necessarily 3-connected. The size of a cubic map is defined as the number of faces minus 2, a convention that simplifies the algebraic computations.

We need the generating function of 3-connected cubic maps, which is related to the generating function $T(z)$ of simple triangulations. The latter was obtained by Tutte [**10**] and is an algebraic function given by

$$T(z) = U(z)\left(1 - 2U(z)\right), \tag{1}$$

where $z = U(z)(1-U(z))^3$, and z marks the number of vertices minus two. As shown in [**10**], the unique singularity of T, coming from a branch point, is located at $\tau = 27/256$ and $T(\tau) = 1/8$. The singular expansion of $T(z)$ near τ is

$$T(z) = \frac{1}{8} - \frac{3}{16}Z^2 + \frac{\sqrt{6}}{24}Z^3 + O(Z^4),$$

where $Z = \sqrt{1 - z/\tau}$. Notice that τ is a finite singularity, since $T(\tau) = 1/8 < \infty$.

The generating function $M_3(z)$ of 3-connected cubic maps, where z marks the number of faces minus 2 is equal to

$$M_3(z) = T(z) - z. \tag{2}$$

This follows directly from the duality between cubic maps and triangulations, which exchanges vertices and faces.

Adapting directly the proof from [**8**] for cubic planar graphs, one finds that cubic maps are partitioned into five subclasses, as defined below, and where st denotes the root edge of a cubic map M.

- $\mathcal{L}$ (Loop). The root edge is a loop.
- $\mathcal{I}$ (Isthmus). The root edge is an isthmus (an alternative name for a bridge).
- $\mathcal{S}$ (Series). $M - st$ is connected but not 2-connected.
- $\mathcal{P}$ (Parallel). $M - st$ is 2-connected but $M - \{s, t\}$ is not connected.
- $\mathcal{H}$ (polyHedral). M is obtained from a 3-connected cubic map by possibly replacing each non-root edge with a cubic map whose root edge is not an isthmus.

3 The Ising model on rooted triangulations and perfect matchings in cubic maps

Given a graph G, its Ising partition function is defined as follows. Given a 2-coloring, not necessarily proper, $c\colon V(T) \to \{1,2\}$ of the vertices of G, let $m(c)$ be the number of monochromatic edges in the coloring. Then define

$$p_G(u) = \sum_{c\colon V(T)\to\{1,2\}} u^{m(c)}.$$

The same definition applies for rooted maps, using the fact that in a rooted map the vertices are distinguishable.

Suppose G is a triangulation with $2n$ faces. Since in a 2-coloring every face of T has at least one monochromatic edge, the number of monochromatic edges at least n. The lower bound can be achieved taking the dual edge-set of a perfect matching in a cubic map. We show next that perfect matchings of a cubic map M with $2n$ vertices are in bijection with 2-colorings of the dual triangulation M^* with exactly n monochromatic edges, in which the color of the root vertex is fixed.

Lemma 3 *Let M be a rooted cubic map and $T = M^*$ its dual triangulation. There is a bijection between perfect matchings of M and 2-colorings of T with exactly n monochromatic edges in which the color of the root vertex of T is fixed.*

The generation function of the Ising partition of triangulations is defined as

$$P(z,u) = \sum_{T\in\mathcal{T}} p_T(u) z^n,$$

where $\mathcal{T}$ is the class of rooted triangulations and the variable z marks the number of vertices minus 2. An expression for P was obtained by Bernardi and Bousquet-Mélou [**2**] in the wider context of counting q-colorings of maps with respect to monochromatic edges, which is equivalent to computing the q-Potts partition function. It is the algebraic function $Q_3(2,\nu,t)$ in [**2**, Theorem 23]. Here the parameter 2 refers to the number of colors, t marks edges and ν marks monochromatic edges. Extracting the coefficient $[\nu^n]Q_3(2,\nu,t)$ we obtain a generating function which is equivalent to the generating function $M(z)$ of rooted cubic maps with a distinguished perfect matching, where z marks faces minus 2. After a simple algebraic manipulation we obtain:

Lemma 4 *The generating function $M = M(z)$ counting rooted cubic maps with a distinguished perfect matching satisfies the quadratic equation*

$$72\, M^2 z^2 + \left(216\, z^2 - 36\, z + 1\right) M + 162\, z^2 - 6\, z = 0. \tag{3}$$

where the variable z marks the number of faces minus two.

The former quadratic equation has a non-negative solution

$$M(z) = \frac{-1 + 36z - 216z^2 + (1-24z)^{3/2}}{144z^2}.$$

Expanding the binomial series one obtains the closed formula

$$[z^n]M(z) = 3\cdot 6^n \frac{\binom{2n}{n}}{(n+2)(n+1)}, \tag{4}$$

a formula which can be proved combinatorially [**9**].

4 From the Ising model to 3-connected cubic graphs

We use the decomposition of cubic graphs as in [**3**] and [**8**], and the following observation. We say that a class $\mathcal{N}$ of rooted maps is closed under rerooting if whenever a map N is in $\mathcal{N}$, so is any map obtained from N by forgetting the root edge and choosing a different one.

Lemma 5 *Let $\mathcal{N}$ be a class of cubic maps closed under rerooting with a distinguished perfect matching. Let $\mathcal{N}_1$ be the maps in $\mathcal{N}$ whose root edge belongs to the perfect matching, and $\mathcal{N}_0$ those whose root edge does not belong to the perfect matching. Let $N_i(z)$ be the associated generating functions. Then $N_0(z) = 2N_1(z)$.*

The previous lemma applies in particular to the class of all cubic maps and to the class of 3-connected cubic maps.

Lemma 6 *The following system of equations holds and has a unique solution in power series with non-negative coefficients.*

(5)

$$\begin{array}{llll}
M_0(z) & = D_0(z), & M_1(z) & = D_1(z) + I(z), \\
D_0(z) & = L(z) + S_0(z) + P_0(z) + H_0(z), & D_1(z) & = S_1(z) + P_1(z) + H_1(z) \\
I(z) & = \frac{L(z)^2}{4z}, & L(z) & = 2z(1 + D_0(z)) \\
S_1(z) & = D_1(z)(D_1(z) - S_1(z)), & S_0(z) & = D_0(z)(D_0(z) - S_0(z)) \\
P_1(z) & = z(1 + D_0(z))^2, & P_0(z) & = 2z(1 + D_0(z))(1 + D_1(z)), \\
H_1(z) & = \dfrac{T_1(z(1 + D_1(z))(1 + D_0(z))^2)}{1 + D_1}, & H_0(z) & = \dfrac{T_0(z(1 + D_1(z))(1 + D_0(z))^2)}{1 + D_0}.
\end{array}$$

We sketch the justification of the former equations, starting with an observation. An edge e is replaced with a map whose root edge is in a perfect matching if and only if the two new edges resulting from the subdivision and replacement of e belong to the resulting perfect matching. The equation for $I(z)$ is because an isthmus map is composed of two loop maps; division by 4 takes into account the possible rootings of the two loops. The situation for $L(z)$, $S_i(z)$, $P_i(z)$ and $H_1(z)$ are rather straightforward. The equations for H_i can be detailed as follows: in a cubic map with $2n$ vertices there are n edges in a perfect matching and $2n$ not in it, hence the term $(1 + D_1(z))(1 + D_0(z))^2$ in the substitution.

By elimination we obtain $T_1(z)$ and $T_0(z) = 2T_1(z)$. The equation defining T_1 is

$$\begin{aligned}
&T_1^6 + (24\,z + 16)\,T_1^5 + \left(60\,z^2 + 92\,z + 25\right)T_1^4 + \left(80\,z^3 + 208\,z^2 + 96\,z + 19\right)T_1^3 \\
&+ \left(60\,z^4 + 232\,z^3 + 150\,z^2 + 12\,z + 7\right)T_1^2 + \left(24\,z^5 + 128\,z^4 + 112\,z^3 + z^2 - 16\,z + 1\right)T_1 \\
&+4\,z^6 + 28\,z^5 + 33\,z^4 + 12\,z^3 - z^2 = 0.
\end{aligned}$$

5 From 3-connected cubic maps to cubic planar graphs

A cubic *network* is a connected cubic planar multigraph G with an ordered pair of adjacent vertices (s, t) such that the graph obtained by removing one of the edges between s and t is connected and simple. We notice that st can be a simple edge, a loop or be part of a double edge, but cannot be an isthmus. The oriented edge st is called the *root* of the network, and s, t are called the *poles*. Replacement in networks is defined as for maps. We let $\mathcal{D}$ be the class of cubic networks. The classes $\mathcal{I}$, $\mathcal{L}$, $\mathcal{S}$, $\mathcal{P}$ and $\mathcal{H}$ have the same meaning as for maps, and so do the subindices 0 and 1. We let $\mathcal{C}$ be the class of connected cubic planar graphs (always with a distinguished perfect matching), with associated generating function $C(x)$, and $C^\bullet(x) = xC'(x)$ be the generating functions of those graphs rooted at a vertex. We also let $G(x)$ be the generating function of arbitrary (non-necessarily connected) cubic planar graphs.

Whitney's theorem claims that a 3-connected planar graph has exactly two embeddings in the sphere up to homeomorphism. Thus counting 3-connected planar graphs rooted at a directed edge amounts to counting 3-connected maps, up to a factor 2. Below we use the notation $T_i(x)$ for the exponential generating functions of 3-connected cubic planar graphs rooted at a directed edge, similarly to maps.

Lemma 7 *The following system of equations holds and has a unique solution in power series with non-negative coefficients.*

$$(6)\qquad \begin{array}{llll} D_0 &= L + S_0 + P_0 + H_0, & D_1 &= S_1 + P_1 + H_1 \\ I &= \frac{L^2}{x^2}, & L &= \frac{x^2}{2}(D_0 - L) \\ S_1 &= D_1(D_1 - S_1, 6, & S_0 &= D_0(D_0 - S_0 \\ P_1 &= x^2 D_0 + \frac{x^2}{2} D_0^2, & P_0 &= x^2(D_0 + D_1) + x^2 D_0 D_1 \\ H_1 &= \dfrac{T_1(x^2(1+D_1)(1+D_0^2))}{2(1+D_1)}, & H_0 &= \dfrac{T_0(x^2(1+D_1)(1+D_0^2))}{2(1+D_0)}. \end{array}$$

Moreover, we have

$$3C^\bullet = I + D_0 + D_1 - L - L^2 - x^2(D_0 + D_1) - x^2 D.$$

6 Proofs of the main results

Proof of Theorem 2. We first need to find the dominant singularity of $C(x)$, which is the same as that of $D_0(x)$, $D_1(x)$ and then $D(x)$. It is obtained by first computing the minimal polynomial for $D(x)$ and then its discriminant $\Delta(x)$. After discarding several factors of $\Delta(x)$ for combinatorial reasons (as in [**8**]), the relevant factor of $\Delta(x)$ turns out to be

$$904x^8 + 7232x^6 - 11833x^4 - 45362x^2 + 3616,$$

whose smallest positive root is equal to $\sigma \approx 0.27964$. After routinely checking the conditions of [**8**, Lemma 15], we conclude that σ is the only positive dominant singularity and that $D(x)$ admits an expansion near σ of the form

$$D(x) = d_0 + d_2 X^2 + d_3 X^3 + O(X^4), \qquad X = \sqrt{1 - x/\sigma}.$$

And the same hold for $D_0(x)$ and $D_1(x)$. But also for $L(x)$ and $I(x)$, using their definitions given in terms of $D_0(x)$ in Lemma 6. There is a second singularity $-\sigma$ with a similar singular expansion and, as explained in [**8**], the contributions of $\pm\sigma$ are added.

From there, and using again Lemma 6 we can compute the singular expansion of $C^\bullet(x) = xC'(x)$, and by integration, that of $C(x)$. For arbitrary cubic planar graphs, we use the exponential formula $G(x) = e^{C(x)}$, which encodes the fact that a graph is an unordered set of connected graphs. The transfer theorem finally gives

$$(7)\qquad G_n = [x^n]G(x) \approx c_1 n^{-7/2} \sigma^{-n} n!.$$

To obtain the expected value of X_n we have to divide G_n by the number g_n of labeled cubic planar graphs, which as shown in [**3, 8**] is asymptotically $g_n \sim c_0 n^{-7/2} \rho^{-n} n!$, where $c_0 > 0$ and $\rho \approx 0.31923$ is the smallest positive root of

$$729x^{12} + 17496x^{10} + 148716x^8 + 513216x^6 - 7293760x^4 + 279936x^2 + 46656 = 0.$$

And we obtain the claimed result by setting $c = c_1/c_0$ and $\delta = \rho/\sigma$. Furthermore, since σ and ρ are algebraic numbers, so is δ (actually of degree 48).

For the second part of the statement we argue as follows. Since a graph with n vertices has at most $n!$ automorphisms, the number of unlabeled graphs in a class is at least the number of labeled graphs divided by $n!$. It follows that the number U_n of unlabeled cubic planar graphs with a distinguished perfect matching is at least $G_n/n!$, where G_n is given in (7).

No precise estimate is known for the number u_n of unlabeled cubic planar graphs, but it can be upper bounded by the number C_n of simple rooted cubic planar *maps*, because a planar graph has at least one embedding in the plane. These maps have already been counted in [**6**] and the estimate $C_n \sim c_s \cdot n^{-5/2}\alpha^{-n}$, where $\alpha \sim 0.3102$, follows from [**6**, Corollary 3.2]. The relation between α and the value x_0 given in [**6**] is $\alpha = x_0^{1/2}$; this is due to the fact that we count cubic maps according to faces whereas in [**6**] they are counted according to vertices, and a map with $n+2$ faces has $2n$ vertices. Disregarding subexponential terms, we have $U_n \geq \sigma^{-n}$ and $u_n \leq \alpha^{-n}$. The last result holds as claimed since $\alpha/\sigma \approx 1.109$. □

Proof of Theorem 1. The proof follows the same scheme as that of Theorem 2 and is omitted. One just needs to adapt the system (6) to bridgeless cubic planar graphs by removing the generating functions $I(z)$ and $L(z)$, and follow a similar procedure.

7 Concluding remarks

A natural open question is to prove some kind of concentration result for the number of perfect matchings in cubic planar graphs. But already computing the variance seems out of reach with our techniques, since for computing the second moment we would need to consider maps or graphs with a pair of distinguished perfect matchings, and this does not seem feasible using the connection with the Ising model on triangulations.

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