Problem proposals

5th GAPCOMB Workshop

9 – 12 July 2024, Montserrat, Catalunya, Spain

Contents

1 Quantum projective planes

Communicated by Simeon Ball.

Keywords *Latine squares, projective geometry*

1.1 Background

A *quantum latin square* of order n is a set $\{|\phi_{ij}\rangle \mid i,j \in \{1,\dots,n\}\}$ of n^2 states in $\mathbb{C}^n$ with the property that for all $i \in \{1,\dots,n\}$, $\{|\phi_{ij}\rangle \mid j \in \{1,\dots,n\}\}$ is an orthonormal basis of $\mathbb{C}^n$ and for all $j \in \{1,\dots,n\}$, $\{|\phi_{ij}\rangle \mid i \in \{1,\dots,n\}\}$ is an orthonormal basis of $\mathbb{C}^n$. Two quantum latin squares are *orthogonal* if furthermore $\{|\phi_{ij}\rangle \otimes |\psi_{ij}\rangle \mid i,j \in \{1,\dots,n\}\}$ is an orthonormal basis of $\mathbb{C}^n \otimes \mathbb{C}^n$.

We define a *quantum projective plane* of order n as a set of n^2+n+1 states $\{|\ell_i\rangle \mid i \in \{1,\dots,n^2+n+1\}\}$ of $\mathbb{C}^{n^2+n+1}$ with the property that $\langle \ell_j|\ell_i\rangle = 1 + n\delta_{ij}$. Observe that we can make a quantum projective plane of n from a classical projective plane of order n by mapping a line $\{p_1,\dots,p_{n+1}\} \subset \{1,\dots,n^2+n+1\}$ to the state $|p_1\rangle + \cdots + |p_{n+1}\rangle$.

We can also make a quantum projective plane of n directly from a set of $n-1$ mutually orthogonal latin squares in the following way. Labelling the cell (i,j) with the ket $|(i-1)n+j\rangle$ we can make lines in a quantum projective plane considering cells where the element $j \in \{1,\dots,n\}$ falls in the latin square. In this way each latin square gives us n lines where we add a state $\left|n^2+r\right\rangle$ if we are using the r-th latin square. The vertical and horizontal parallel lines give $2n$ additional lines and the line at infinity will be the state $|\ell_{n^2+n+1}\rangle = \left|n^2+1\right\rangle + \cdots + \left|n^2+n+1\right\rangle$.

Example 1. *Consider the classical orthogonal latin squares*

$$\begin{array}{|ccc|}\hline 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|ccc|}\hline 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \\ \hline\end{array}$$

We can make two orthogonal quantum latin squares by mapping j to the state $|j\rangle$,

$$\begin{array}{|ccc|}\hline |1\rangle & |2\rangle & |3\rangle \\ |2\rangle & |3\rangle & |1\rangle \\ |3\rangle & |1\rangle & |2\rangle \\ \hline \end{array} \qquad \begin{array}{|ccc|}\hline |1\rangle & |2\rangle & |3\rangle \\ |3\rangle & |1\rangle & |2\rangle \\ |2\rangle & |3\rangle & |1\rangle \\ \hline \end{array}$$

Following the recipe above we get the 13 lines defining a quantum projective plane to be

$$|\ell_1\rangle = |1\rangle + |6\rangle + |8\rangle + |10\rangle\,, |\ell_2\rangle = |2\rangle + |4\rangle + |9\rangle + |10\rangle\,,$$

$$|\ell_3\rangle = |3\rangle + |5\rangle + |7\rangle + |10\rangle\,, |\ell_4\rangle = |1\rangle + |5\rangle + |9\rangle + |11\rangle\,,$$

$$|\ell_5\rangle = |2\rangle + |6\rangle + |7\rangle + |11\rangle\,, |\ell_6\rangle = |3\rangle + |4\rangle + |8\rangle + |11\rangle\,,$$

$$|\ell_7\rangle = |1\rangle + |4\rangle + |4\rangle + |12\rangle\,, |\ell_8\rangle = |2\rangle + |5\rangle + |8\rangle + |12\rangle\,,$$

$$|\ell_9\rangle = |3\rangle + |6\rangle + |9\rangle + |12\rangle\,, |\ell_{10}\rangle = |1\rangle + |2\rangle + |3\rangle + |13\rangle\,,$$

$$|\ell_{11}\rangle = |4\rangle + |5\rangle + |6\rangle + |13\rangle\,, |\ell_{12}\rangle = |7\rangle + |8\rangle + |9\rangle + |13\rangle\,,$$

$$|\ell_{13}\rangle = |10\rangle + |11\rangle + |12\rangle + |13\rangle\,.$$

Example 2. *Consider the following quantum latin square of order 4.*

$$\begin{array}{|cccc|}\hline |1\rangle & |2\rangle & |3\rangle & |4\rangle \\ \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) & \frac{1}{\sqrt{5}}(i\,|1\rangle + 2\,|4\rangle) & \frac{1}{\sqrt{5}}(2\,|1\rangle + i\,|4\rangle) & \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) \\ \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle) & \frac{1}{\sqrt{5}}(2\,|1\rangle + i\,|4\rangle) & \frac{1}{\sqrt{5}}(i\,|1\rangle + 2\,|4\rangle) & \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle) \\ |4\rangle & |3\rangle & |2\rangle & |1\rangle \\ \hline \end{array}$$

See [1] and [2] for more background.

1.2 Open problems

Problem 1. *Now, the question arises if we can make a quantum projective plane from a set of $n-1$ mutually orthogonal quantum latin squares of n.*

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2 Fast geometric construction on a budget

Communicated by Alberto Espuny Díaz.

Keywords *Random geometric graphs, online construction, budget*

2.1 Background

Let $n \in \mathbb{N}$ and $N := \binom{n}{2}$, and consider n mutually independent random variables $X_1, \ldots, X_n \sim \mathcal{U}([0,1]^2)$, that is, n points chosen independently and uniformly at random from the unit square. For each pair $\{i, j\} = e \in \binom{[n]}{2}$, define the random variable $\ell_e := \|X_i - X_j\|$, where $\|\cdot\|$ denotes the Euclidean norm. The random outcomes of the variables $X_1, \ldots, X_n$ result in a (random) ordering $e_1, \ldots, e_N$ of all the pairs in $\binom{[n]}{2}$, where we choose an ordering such that $\ell_{e_1} \leq \ldots \leq \ell_{e_N}$ (note that this ordering is uniquely determined with probability 1). Now one can define a sequence of nested graphs $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_N$, where $G_i = ([n], \{e_j : j \in [i]\})$. This sequence of graphs is known as the *geometric random graph process*, and many of its asymptotic properties have been very well studied. We say that an event holds *asymptotically almost surely* (a.a.s.) if the probability that it holds tends to 1 as $n \to \infty$.

Now consider the following "game". A player, called Builder, wants to construct a graph on $[n]$. She receives edges one by one and, each time she receives an edge, she must choose to either "purchase" the edge and add it to the graph, or let it go (and, in this case, it is gone forever). Builder must make her choice based solely on the information she knows so far, in terms of which edges have been offered to her; she does not know the full order in which the edges will arrive. But she does know what "distribution" the edges respond to. Here, we want to consider the case in which the edges arrive in the order provided by the geometric random graph process (but, crucially, Builder does not know the positions of the points!). Note that the choices made by Builder result in a sequence of nested graphs $B_0 \subseteq B_1 \subseteq \ldots \subseteq B_N$ such that $B_i \subseteq G_i$ for all i.

The goal of Builder is to construct a graph which satisfies some desired (non-trivial, monotone) property $\mathcal{P}$ (e.g., being Hamiltonian). This can be achieved in many ways. For instance, she could choose to purchase every edge, as they arrive, until the graph satisfies the desired property; this will result simply in reproducing the random geometric graph process. The catch is that Builder has a limited budget b for the number of edges that she can purchase in total, that is, she must maintain $|E(B_i)| \leq b$ at all times (and recall she cannot remove edges once purchased!).

Let us use Hamiltonicity as an example: if Builder is allowed a budget of at least $(1/2 + \epsilon)n \log n$ edges, then she can simply run the random geometric graph process purchasing all edges and, by the time she cannot purchase any more edges, a.a.s. her graph will be Hamiltonian (see [2]). If she is allowed a budget of exactly n edges, then one option is to simply pick a Hamilton cycle H beforehand and then run the process, purchasing only those edges that belong to H, but this takes a very long time. In general, given a budget, for how long does Builder have to run the process until she can construct a graph containing a Hamilton cycle with that budget?

2.2 Open problems

For any pair of positive integers t and b, a (t, b)-*strategy* for Builder is a (possibly random) function which, for each $i \in [t]$, given the history of the process so far (that is, the sequence $G_0 \subseteq \ldots \subseteq G_{i-1}$), the graph B_{i-1}, and presented with a new edge at time i, decides whether to purchase the edge, under the limitation that $|E(B_i)| \leq b$.

Problem 2. *For which pairs (t, b) does Builder have a (t, b)-strategy such that a.a.s. $B_t \in \mathcal{P}$?*

The aim with this problem would be to pick one (or a few) simple properties and analyse this question for them. Some examples could be Hamiltonicity, perfect matchings or connectivity, but also the containment of fixed subgraphs, such as triangles, trees, etc.

This question is motivated by an analogous question when the edges presented to Builder are picked uniformly at random among the missing ones. This model was proposed by Frieze, Krivelevich and Michaeli [3] and subsequently studied by other authors [1, 5, 4]; with this problem, we propose the study of a geometric variant where, by virtue of knowing that the edges are given by the random geometric graph process, Builder may be able to come up with different strategies.

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3 Connectivity in hypergraphs

Communicated by Richard Lang.

Keywords *hypergraph, connectivity, minimum degree*

3.1 Background

The following problem is motivated by recent developments in extremal simplicial topology, in particular the work of Georgakopoulos, Haslegrave, Montgomery and Narayanan [2].

A *k-uniform hypergraph* G consists of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge is a set of k vertices. Let G^* be the *line graph* of G, which is the graph on vertex set $E(G)$ with an edge ef whenever $|e \cap f| \leq k-1$. A subgraph of G without isolated vertices is said to be *tightly connected* if its edges induce a connected subgraph in G^*.

3.2 Open problems

Problem 3. *Let G be a 3-uniform hypergraph on n vertices such that every vertex is on more than $\frac{4}{9}\binom{n}{2}$ edges. Show that G contains a tightly connected subgraph on n vertices.*

A few remarks.

- To see that the degree condition cannot be lowered (significantly), consider the 3-uniform hypergraph G whose vertices are partitioned by sets X, Y and Z each of size $n/3$ and whose edges are composed of all edges of type XXY, YYZ and ZZX as well as all edges inside each of X, Y and Z. It is not hard to see that G does not have a vertex-spanning tightly connected subgraph. (In fact, each tightly connected subgraph spans at most two of the parts X, Y and Z.) On the other hand, a simple calculation shows that every vertex of G is on at least $(4/9 - o(1))\binom{n}{2}$ edges.

- It is known that the problem can be solved, if the lower bound on the edges is replaced by $\frac{5}{9}\binom{n}{2}$. This follows from the work of

Cooley and Mycroft [1, Lemma 3.2], who prove a result about 2-graphs that can be applied to the 2-uniform graphs induced by the edges incident to each vertex (so called *link graphs*).

- To get a feeling for the problem, it might be instructive to first solve it under the (stronger) assumption that every pair of vertices is on more than $n/3$ edges. (This is essentially sharp due to the above construction.)

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4 Dynamic interpolation between copies of a graph H in a dense host graph

Communicated by Lyuben Lichev.

Keywords *connectivity thresholds, random walks*

4.1 Background

A main topic in extremal combinatorics is related to finding minimum degree thresholds guaranteeing the existence of certain substructures in a sufficiently dense host graph. One prominent example is Dirac's theorem, which says that every graph on n vertices and minimum degree at least $n/2$ contains a Hamilton cycle. Dirac's theorem has seen a number of generalisations, most notably for powers of cycles and for bounded-degree graphs on n vertices and bandwidth $o(n)$ (the latter breakthrough of Böttcher, Schacht and Taraz confirmed a corresponding conjecture of Bollobás and Komlós).

4.2 Open problems

This project takes a more dynamic point of view on the above problem. More precisely, fix an n-vertex graph H of bounded maximum degree and an n-vertex host graph G above the minimum degree threshold for containment of H. Define $\mathcal{H}$ as the family of copies of H in G and define the relation $H_1 \sim H_2$ between two copies $H_1, H_2 \in \mathcal{H}$ if they can be obtained from each other by exchanging the positions of a pair of vertices. This allows us to see the family $\mathcal{H}$ as a graph.

Problem 4. *What is the minimum degree threshold for connectivity of the graph $\mathcal{H}$ if H is a Hamilton cycle? What about more general graphs of bounded maximum degree and bandwidth $o(n)$?*

Once Problem 4 is answered, a natural question is how far two copies of H could be in $\mathcal{H}$.

Problem 5. *What can be said about the diameter of $\mathcal{H}$ over the connectivity threshold?*

The connectivity of $\mathcal{H}$ is desirable for at least two reasons. First, it allows to interpolate between the copies of H in a "continuous" way via some local operation. Such interpolation is often useful in problems dealing with edge-colourings or edge-weights where a single vertex exchange only barely modifies the characteristics of the observed copy of H. Second, upon some additional regularity constraints for the degrees and the co-degrees of the graph G, it is believeable that a fast sampling algorithm can be derived from the simple random walk on $\mathcal{H}$.

Problem 6. *Upon determination of the threshold in Problem 4, what is the mixing time of the simple random walk on $\mathcal{H}$? When does such a random walk provide us with an algorithm for sampling an (approximately) uniformly random copy of H.*

Of course, an answer to Problem 5 naturally gives a lower bound on the mixing time in Problem 6.

Furthermore, it is natural to believe that binomial random graphs allow fast interpolation between two copies of H even when they contain $o(n^2)$ edges.

Problem 7. *Is there a sharp threshold p_c for connectivity of the auxiliary graph $\mathcal{H}$ in $G(n,p)$? When $p \geq (1+\varepsilon)p_c$, how fast does the random walk on $\mathcal{H}$ mix? For which values of p does this provide a fast algorithm for sampling an (approximately) uniform copy of H in $G(n,p)$?*

5 Random friend trees

Communicated by Dieter Mitsche.

Keywords *random trees*

5.1 Background

The following random recursive tree model was considered by Addario-Berry, Briend, Devroye, Donderwinkel, Kerriou and Lugosi [1]: new vertices are attached in a sequential manner one by one by selecting an existing vertex (called target vertex) in the tree, but instead of connecting to the target vertex by an edge, they connect to one of its neighbors (or friends), chosen uniformly at random. This model gives rise to different behavior than seen both in the random recursive tree as well as in the preferential attachment model: the authors show new results on the number of leaves, on the maximum degree, typical distance and diameter of such a graph, among others.

5.2 Open problems

It is natural to extend the model considered by [1] with different choices of choosing the target vertex, and we suggest the following modifications:

Problem 8. *A first variation of the model is to choose $0 < p < 1$ and connect the new vertex by an edge to the target vertex with probability p, and to a uniformly chosen neighbor of the target vertex with probability $1-p$. This modification makes it much easier for neighbors of high-degree vertices to grow their degree, and in particular, the degree of every vertex goes to infinity almost surely as the tree grows. What can be said about the number of leaves or the maximum degree of such a model?*

Problem 9. *A second variation is the following: after choosing the random target vertex, perform a random walk of length k and connect the new vertex by an edge to the terminal vertex of this random walk. What can be said about the number of leaves or the maximum degree of such a model? One could start with $k = 2$; it is tempting to believe that for even k the number of leaves is higher, and for odd k the maximum degree is higher (performing a random walk of infinite length results again in the preferential attachment model).*

Finally, another question in the original model of [1] is the following:

Problem 10. *What does the forest induced by the vertices of degree at most N, for large N, look like? Do these subtrees look like random friend trees?*

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6 A randomly perturbed Čern'y's conjecture

Communicated by Patrick Morris.

Keywords *Finite automata, synchronizing words, random perturbations*

6.1 Background

The notion of automaton is some important concept in computer science that I don't know much about but there is a very famous conjecture (it's even on wikipedia) of Čern'y that is pretty combinatorial.

The simple object is an *automaton* with n states and two functions $a, b : [n] \to [n]$. By concatenating the functions, you can apply *words* $w : [n] \to$ where w is some string of as and bs. For example, the word $w = aab$ means that you apply b and then a twice. Now the automaton is said to be *synchronizable* if there is *some* word w such that $w(i) = w(j)$ for all $i, j \in [n]$. Such a word is called a synchronizing word.

The conjecture of Čern'y from 1969 states that every synchronizable automaton has some synchronizing word of length at most $(n-1)^2$ (which would be tight due to some simple example). Despite lots of attention, this conjecture remains wide open with the best known bound being of order n^3.

In some recent work of Guillem/aume [1] investigated random automata and showed that with high probability they have much shorter synchronizing words of order $n^{1/2}$ (ignoring polylogs). The problem I suggest is to see what happens to the length of a synchronizing word after randomly perturbing some arbitrary automata.

6.2 Open problems

Problem 11. *Given some arbitrary (synchronizable) automata $a, b \to [n]$, what can be said about the length of a synchronizing word after t random changes? In particular, how many random changes need to be applied until you can guarantee that Čern'ys conjecture holds for the outcome.*

There is some flexibility here on what is meant by 'random changes' as you can change just a or both a and b but in either case, the most natural thing to do I guess is to impose $a(x) = y$ for some randomly chosen x, y. I expect that doing this already a small number of times should give some much better control over synchronizing word lengths.

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7 Hamilton cycles in powers of cycles

Communicated by Tássio Naia.

Keywords *Hamilton cycle, minimum degree condition*

7.1 Background

A long line of research is concerned with minimum degree conditions for a graph to contain a Hamilton cycle. An important category of problems in the area deals with high-minimum-degree subgraphs of structured graphs. This problem is about generalizing Dirac's well known theorem.

Theorem 12. *If G is a graph of order n and minimum degree at least $n/2$, then G contains a Hamilton cycle.*

We may see the theorem above as saying that, if one takes a complete graph K_n and a subgraph $H \subseteq K_n$ with a certain minimum degree, then H is Hamiltonian. The proposed problems are about what happens when we attempt to generalize Dirac's theorem by replacing K_n and adjusting the minimum degree condition.

For any graph G, let G^k denote the k-th power of G (that is to say, the graph with the same vertex set as G, but where each pair of distinct vertices u, v is joined by an edge when the distance between u and v in G is at most k). A Hamilton cycle in a graph G is a path that contains all vertices of G. Let C_n denote a cycle ion n vertices.

7.2 Open problems

The target direction is the following conjecture.

Conjecture 13 (Espuny-Díaz, Lichev, Wesolek [1])**.** *For all integers $n \geq 3$ and $k \in [1, n/2]$, every graph $G \subseteq C_n^k$ with minimum degree at least $k + 1$ contains a Hamilton cycle.*

A good starting point might be showing that such subgraphs contain large cycles.

Problem 14. *Show that there exists $\alpha > 0$ such that, for all large n, k, every subgraph of C_n^k with minimum degree $k+1$ contains a cycle of length at least αn.*

As the conjecture is quite recent, there is a lot of flexibility here. It might be useful to consider requiring larger minimum degree.

Problem 15. *Show that for some $\ell \geq 1$ and sufficiently large k, there exists α, > 0 such that, for all large n, k, every subgraph of C_n^k with minimum degree $k+\ell$ contains a cycle of length at least αn.*

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8 Three points in parallel line

Communicated by Tássio Naia.

Keywords *Plane geometry, points and lines, parallell*

8.1 Background

This is a simple-looking problem proposed by Ruzsa at the *Additive Combinatorics and Fourier Analysis Workshop*[1] (17/jun–21/jun 2024). There is no clear history about this problem.

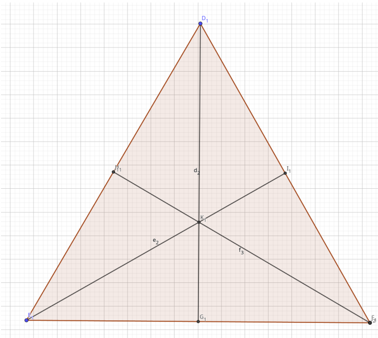
Consider a set $\mathcal{C}$ of points in $\mathbb{R}^2$ not all colinear, that satisfy the following property: for all distinct P, $Q \in \mathcal{C}$, there exists at least one line

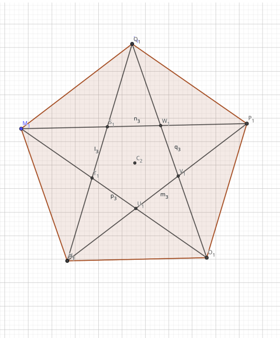
[1] https://erdoscenter.renyi.hu/events/additive-combinatorics-and-fourier-analysis-workshop

in $\mathbb{R}^2$ that contains at least 3 points in $\mathcal{C}$ **and** is parallell to the segment PQ. We say that such a line satisfies the 3-point parallel property for the pair $\{P, Q\}$.

Two examples of such finite configurations are the following.

- The 7-point collection formed by the vertices of a triangle T, the midpoints of the sides of T and its baricenter.
- The 11 point collection formed by the intersections of the segments forming a regular pentagon's sides and its diagonals, plus the pentagon's baricenter.





8.2 Open problems

Problem 16. *Is there any other configuration of points that works?*

What if we relax the condition to requiring only that some (fraction?) of pairs of points in $\mathcal{C}$ admit such a parallel?

Problem 17. *Fix an integer ℓ. What finite point collections in $\mathbb{R}^2$ are such that all but at most ℓ pairs of distinct P, $Q \in \mathcal{C}$ have the 3-point parallel property.*

9 Rainbow path covers

Communicated by Tássio Naia.

Keywords *Rainbow graphs, graph cover, paths, edge-coloring*

9.1 Background

In a recent paper [2], it has been shown that most properly edge-colored graphs can be covered by a linear number of rainbow paths.
Definitions. A *properly edge-colored* graph contains no monochromatic path of length 2, and a path is said to be rainbow if no two edges in it have the same color. A collection $\mathcal{P}$ of paths of a graph G is a path cover if every edge of G appears in some path in $\mathcal{P}$.

Theorem 18. *Let $G = G(n,p)$ be the binomial random graph. If $p \gg (\log n/n)^2$, then asymptotically almost surely every proper coloring of G admits a rainbow path cover.*

This is the first progress towards the following conjecture.

Conjecture 19 ([1])**.** *Given a graph G of order n and a proper edge-coloring of G, there exists a rainbow path cover of G formed by $\mathrm{O}(n)$ paths.*

For convenience, let us call a graph G *linearly rainbow coverable* if G satisfies conjecture 19.

9.2 Open problems

There at least two clear directions of research for this problem. The first one is extending the results for other classes of graphs, such as graphs of large minimum degree, or d-regular graphs where $d = d(n) = \omega(1)$ is a slowly growing function of n.

Problem 20. *Find a new class of graphs (such as graphs with large minimum degree) which is linearly rainbow coverable.*

The second avenue is strengthening known results by considering rainbow path *decompositions.*

Problem 21. *Find families of graphs G that have the following property: for each proper edge-coloring of G, there exists a partition of the edges of G into $\mathrm{O}\big(|V(G)|\big)$ rainbow paths.*

There are many further possible variants and strengthenings of this problem. One may consider almost covers/decompositions, or may allow edges to be covered up to a certain number times, or allow a few edges to be missed in $\mathcal{P}$.

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10 Proper conflict-free colourings of random graphs

Communicated by Guillem Perarnau.

Keywords *Colourings, random graphs.*

10.1 Background

In recent years there has been interest in proper colourings that have some additional condition on the number of times a colour appears in the neighbourhood of each vertex. Examples are frugal or dynamical colorings, or colorings of a graph power. On the other hand, the notion of conflict-free colouring was introduced by Pach and Tardos [3], in the context of hypergraphs. The notion admits a natural translation to graphs: we say that a vertex-colouring is *conflict-free* if for every vertex there exists at least one colour that appears exactly once in its neighbourhood. Glebov, Szabó and Tardos [2] studied the evolution of the conflict-free chromatic number, that is the minimum number of colours of a conflict-free colouring, in an Erdős-Rényi random graph $\mathbb{G}(n,p)$, showing that if $p \geq 1/2$, then it differs from the dominating number by at most 3.

10.2 Open problems

Proper conflict-free colourings were introduced by Fabrici et al. [1], and are proper colourings that are also conflict-free. The question proposed here is to study the proper conflict-free chromatic number χ_{cf} of a random graph for any p that is large enough. It is interesting to understand what is the (exact) value of that number.

Problem 22. *For constant $p > 1/2$, is it true that*

$$\chi_{cf} = \big(\chi(\mathbb{G}(n,p)) + O(\gamma(\mathbb{G}(n,p))) \text{ or } \chi_{cf} = \big(\chi(\mathbb{G}(n,p)) + o(\gamma(\mathbb{G}(n,p))).$$

Can you give more precise results, such as concentration on a constant number of points? Understanding better this parameter would give insight on how "random" are proper colourings of a random graph with an (almost) optimal number of colours.

This problem is motivated by discussions on the topic with Bruce Reed and Liana Yepremyan.

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11 Locally identifying colouring of chordal graphs

Communicated by Clément Requilé.

Keywords *Locally identifying colourings, chromatic number, chordal graphs*

This problem focuses on vertex-colourings allowing to distinguish the vertices of a graph. Let $N[u]$ denote the closed neighbourhood of a vertex u in a graph. A *locally identifying colouring*, or *lid-colouring*, of a graph is a proper colouring c such that, for any edge uv if $N[u] \neq N[v]$ then $c(N[u]) \neq c(N[v])$. The *lid-chromatic number* $\chi_{\text{lid}}(G)$ of a graph G is the minimum number of colours used in a lid-colouring of G.

In [1] the authors study the parameter the lid-chromatic number for different families of graphs, and made some conjectures several of which have been solved.

One remains however unsolved, and concerns chordal graphs. A graph is said to be chordal if it admits no induced cycle of length four or greater.

Problem 23. *For any chordal graph G, $\chi_{lid}(G) \leq 2\chi(G)$.*

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12 Modular number of perfect matchings in random regular bipartite graphs

Communicated by Clément Requilé.

Keywords *perfect matchings, random graphs*

The following problem was conjectured by Louis Esperet in [1] and is motivated by an extension of the Berge-Sauer conjecture, which stated that every 4-regular graph contains a 3-regular subgraph and was proved by Tashkinov in the 80s.

Problem 24. *For any integer $q \geq 3$ there is a real $\varepsilon > 0$ such that for any $i \in \{0, 1, \ldots, q-1\}$ and n sufficiently large, the number of perfect matchings modulo q of an n-vertex random q-regular bipartite graph is equal to i with probability at least ε.*

The problem can and perhaps should also be studied for multigraphs.

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13 Revisting the asymptotic enumeration of lattices

Communicated by Juanjo Rué.

Keywords *lattices, asymptotic enumeration, hypergraph containers*

13.1 Background

Nowadays the technique of hypergraph containers is a well established method in extremal combinatorics. Its origins go back to the celebrated paper due to Kleitman and Winston ‘’On the number of graphs without 4-cycles” (published in Discrete Mathematics in 1980), where the authors get bounds for the number of labelled graphs on $[n]$ vertices without cycles of length 4 as subgraphs.

However, in the same year there is a paper (due to the same authors) where there appear also some of these seminal ideas. In the paper "The asymptotic number of lattices" (published in Annals of Combinatorics in 1980) the authors prove the following: let us denote by $L(n)$ the number of lattices over $n+2$ elements. Kleitman and Winston, by applying a convenient algorithm, obtained the upper bound

$$L(n) \leq \alpha^{n^{3/2}+o(n^{3/2})},\ \alpha \approx 6.1134.$$

This bound is obtained by reducing the problem to obtaining good upper bounds for lattices with exactly 4 levels on $n+2$ points, which we denote by $S(n)$. More precisely, the authors obtain that

$$S(n) < \beta^{n^{3/2}+o(n^{3/2})},\ \beta \approx 1.6994.$$

This problem is strongly related to the study of the number of 4-cycle free graphs: for a 4-level lattice it is not possible to have 4-cycles defined pairs of pairs of elements between the middle levels.

13.2 Open problems

Problem 25. *Rediscover the bound for $S(n)$ (maybe with an slightly worse constant) by using the modern approach to graph containers. In order to do so, the best is to follow the argument developed by Samotij in [2].*

Once this problem is solved, I also propose the following new improvement:

Problem 26. *Improve the constant β (and consequently α). In order to do that, one needs to understand the improvement on the constant in the case of graphs without 4-cycles due to Balogh and Wagner in [1] (available in Jozsi Balogh's webpage).*

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14 Planar point set configurations

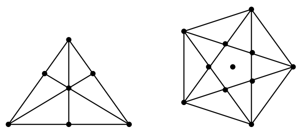
Communicated by Oriol Serra.

Keywords *finite plane geometry, configurations of points and lines*

14.1 Background

This is a simple striking problem posed by Imre Ruzsa at the Workshop on Additive and Analytic Combinatorics held in Budapest in June 2024.

The following two examples are sets of points in the plane with the property that, for every two points there is a line parallel to the one defined by them which contains at least three points of the set.



14.2 Open problems

The problem is simple:

Problem 27. *Let P be a finite set of points in the plane not all in a line with the property that, for every two points $x, y \in P$ there is a line parallel to the line xy which contains three points of P. Are there additional examples besides the ones depicted in the figure? An infinite family?*

15 Long arithmetic progressions in $A - A$

Communicated by Oriol Serra.

15.1 Background

Let A be a set of integers. It is an old question to determine the maximum length of an arithmetic progression contained in sumsets of A. If $|A+A| < 3|A|-4$ then it is known that $A+A$ contains an arithmetic progression of length at least $2|A|-1$. Szemerédi and Vu [Long arithmetic progressions in sumsets: thresholds and bounds. J. Amer. Math. Soc.19 (2006), no.1, 119–169] show that, for any set A, the iterated sumset lA contains an arithmetic progression of length $cl|A|$ whenever $|A| \geq Cn/l$, for some constants c and C.

15.2 Open problems

Illya Shkredov asks the following striking questions:

Problem 28. *Is it true that there is an absolute constant c such that, for every set of integers A, the difference set $A - A$ contains an arithmetic progression of length $c|A|$? If true, is this constant $c \geq 1/2$? Is there some absolute constant l suc that $lA - lA$ contains an arithmetic progression of length $|A|/2$? Is it true that the longest arithmetic progression P in $A - A$ is symmetric, $P = -P$?*

From all these questions perhaps the one about the longest arithmetic progression being symmetric is the easiest oune and the answer is most likely yes. As for the other ones, some computer experiments seem to confirm that answer might be positive for all of them.

16 New strictly Neumaier graphs

Communicated by Robin Simoens.

Keywords *Latin square, Neumaier graph*

16.1 Background

Definition 1. *A graph is edge-regular if there is a constant λ such that every two adjacent vertices have λ common neighbours.*

A graph is co-edge-regular if there is a constant μ such that every two nonadjacent vertices have μ common neighbours.

A graph is strongly regular if it is regular, edge-regular and co-edge-regular.

A regular clique is a clique for which there is a constant e such that every vertex outside the clique has exactly e neighbours in the clique.

Definition 2. *A Neumaier graph is a regular, edge-regular graph with a regular clique.*

A strictly Neumaier graph is a Neumaier graph that is not strongly regular.

People are interested in constructions of strictly Neumaier graphs and their possible parameters.

16.2 Open problems

In Section 4 of [1], a strictly Neumaier graph with parameters $(25, 12, 5; 2, 5)$ was constructed using a Latin square of order 5.

Problem 29. *Can we construct strictly Neumaier graphs using Latin squares of order at least 6?*

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17 Dirac's theorem within powers of cycles

Communicated by Robin Simoens.

Keywords *Hamiltonian graph, graph power*

17.1 Background

Source of the problem: Ramon Llull prize talk of Alberto during the DMD2024.

17.2 Open problems

Conjecture 30. *If $H \subseteq C_n^k$ with $\delta(H) \geq k+1$, then H is Hamiltonian.*

Here, C_n^k denotes the k-th power of the n-cycle and $\delta(H)$ is the minimum degree of H.

Problem 31. *Solve the conjecture for as many values of k.*

18 A Stable Sensitivity Conjecture

Communicated by Christoph Spiegel.

Keywords *Extremal graph theory, Boolean Functions, Fourier analysis*

18.1 Background

We are interested in the relation of different complexity measures of a given boolean function $f : \{-1,1\}^n \to \{-1,1\}$. Given $x \in \{-1,1\}^n$ and $B \subseteq [n]$, let x^B denote the vector denoted by flipping all bits in B.

- Its *sensitivity* denotes the maximum number of bits that can be flipped while changing the output, that is

$$\mathrm{s}(f) = \max_x |\{i \mid f(x) \neq f(x^{\{i\}})\}|.$$

- Its *block sensitivity* denotes the maximum number of disjoint blocks $B_1, \ldots, B_t$ that can be flipped while changing the output, that is

$$\mathrm{bs}(f) = \max_x \max\{t \mid \exists B_1, \ldots, B_t \subset [n] \text{ s.t. } f(x) \neq f(x^{B_i})\}.$$

- Its *degree* is defined as the degree of its unique polynomial expression in which each variable has degree at most 1. Equivalently, it is the cardinality of the largest $I \subseteq [n]$ for which the Fourier transform $\hat{f}(I)$ w.r.t. the polynomial basis given by $\prod_{i \in I} x_i$ is non-zero, that is

$$\deg(f) = \max_I \{|I| \mid \hat{f}(I) \neq 0\}.$$

.

Clearly $\mathrm{bs}(f) \geq \mathrm{s}(f)$ and the *Sensitivity Conjecture* stated that $\mathrm{bs}(f) \leq \mathrm{s}(f)^C$ for some $C > 0$. Since the block sensitivity was already known to be polynomially upper bounded by the degree, it was sufficient to show that the degree is polynomially upper bounded by the sensitivity.

Gotsman and Linial [2] showed that establishing $h(\deg(f)) \leq s(f)$ for any monotonous function h is equivalent to showing that any subset S of the vertices of the n-dimensional hypercube of size at least $|S| > 2^{n-1}$ induces a subgraph with maximum degree at least $h(n)$. This was proven by Huang [3] in 2019 for $h(x) = \sqrt{x}$. Knuth [4] later gave a slightly simplified proof.

18.2 Open problems

Huang's result is tight, as shown by Chung, Füredi, and Graham [1], who constructed a family of sets of size $2^{n-1}+1$ with maximum degree strictly less than $\sqrt{n}+1$.

Problem 32. *Can we establish some form of stability or exactness result in Huang's proof of the Sensitivity Conjecture w.r.t. to the family of sets introduced by Chung, Füredi, and Graham? Alternatively, can we extend the set of families for attaining that bound?*

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19 Crossing numbers of flip sequences

Communicated by Alexandra Wesolek.

Keywords *Flip graphs, non-crossing trees*

19.1 Background

Let C be a set of n points in the plane in convex position. A spanning tree T on C is *non-crossing* if every pair of edges of T (represented by the straight line interval between their endpoints) are pairwise non-crossing. A *flip* on a non-crossing tree T consists of removing an edge e from T and adding another edge f so that the resulting graph $(T \cup f) \setminus e$ is also a non-crossing spanning tree. A *flip sequence* is a sequence of non-crossing spanning trees such that consecutive spanning trees in the sequence differ by exactly one flip. Recently, there has been much interest in finding flip sequences of small length. Optimally one would like to

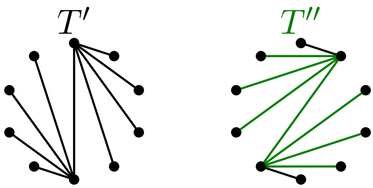

Figure 19.1: Two non-crossing spanning trees between which a flip sequence is long.

find a flip sequence of length $|E(T') \setminus E(T'')|$ between any two trees T', T''. However, this is not always possible, as Hernando et al. showed in 1999 [2]. One needs $\frac{3}{2}n - 5$ flips for the trees in Figure 19.1 as any of the non-common edges in T' is crossed by $\frac{n}{2} - 1$ edges of T'' (and vice versa). Very recent progress showed that one can always find a flip-sequence between two trees of length $\frac{5}{3}n$ (Kleist et al., on the arXiv this week).

19.2 Open problems

I propose a twist on the problem. A *matroid flip* on a tree T consists of removing an edge e from T and adding another edge f so that the resulting graph $(T \cup f) \setminus e$ is a tree (so as before, just dropping the non-crossing condition on the trees). For a spanning tree T on C we denote $cr(T)$ as the number of edge crossings in the tree T. Given two non-crossing trees T', T'' we would like to find a (matroid) flip-sequence $T' = T_1, T_2, \ldots, T_k = T''$ with $k = |E(T') \setminus E(T'')|$ such that $\max_{i \in [k]} cr(T_i)$ is minimized. We denote the minimum as $cr(T', T'')$.

Problem 33. *For any two non-crossing trees T', T'', how large can $cr(T', T'')$ be?*

I would suggest to start by analyzing the trees in Figure 19.1.

A related but quite different problem I would like to propose is on random flips and mixing times.

Problem 34. *What is the mixing time for the flip walk of non-crossing spanning trees on C?*

The same question has been studied for the flip walk for triangulations on C. Similarly as before, a flip on a triangulation R of C consists of removing an edge e from R and adding another edge f so that the resulting graph $(R \cup f) \setminus e$ is a triangulation of C. Eppstein and Frishberg [1] recently improved the upper bound on the mixing time for triangulations for the first time in 25 years from an $O(n^5 \log n)$ given by McShine and Tetali [3] to $O(n^3 \log^3(n))$. To my knowledge, not much is known for the flip walk on non-crossing trees, and I think it would be nice to find some polynomial upper bound on the mixing time. In fact, the first polynomial upper bound on the mixing time for the triangulation flip walk was $O(n^{25})$ given by Molloy, Reed and Steiger [4] and as a first step we could aim for a similar bound.

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Composition of the groups

Group 1
Simeon Ball,
Aldo Kiem,
Xavier Povill,
Lluís Vena.

Group 2
Alejandro Basilio,
Anna de Mier,
Miquel Ortega,
Guillem Perarnau.

Group 3
Jordi Castellví,
Dieter Mitsche,
Robin Simoens,
Alexandra Wesolek.

Group 4
Lyuben Lichev,
Tássio Naia,
Marc Noy,
Tabriz Popatia.

Group 5
Sofiya Burova,
Alberto Espuny,
Christoph Spiegel,
Enric Ventura.

Group 6
Richard Lang,
Amanda Montejano,
Clément Requilé,
Oriol Serra.